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Contents

Introduction

This monograph deals with certain aspects of the general theory of systems. We develop the ergodic theory (i.e., the theory of quasiinvariant and invariant measures) in such infinite-dimensional vector spaces which appear as models of various (physical, economic, genetic, linguistic, social,etc.) processes. The methods of ergodic theory are successfully applied to study properties of such systems. For example, the problem of measurability is directly connected with notions and methods of the theory of games (in this context, see, e.g., the Steinhauss-Mycielski determinateness axiom about the existence of winning strategy [81], Banach-Masur's infinite game on the real axis [124] and so on). The foundation of the ergodic theory was stimulated by the necessity of a consideration of the problem of statistical mechanics and was directly connected with works of G.Birkhoff (1943), Kryloff and Bogoliuboff (1946), E.Hoph (1949) and other famous mathematicians.

Our XXI century can be called a century which intensively applies improvements in the mathematical theory of information transmission. In this theory the following one-dimensional linear stochastic system

$$\xi(t,\omega) = \theta(t) + \Delta(t,\omega)$$

is under consideration, where $t \in T \subseteq R$, H is any topological vector space,Θ is a vector subspace of H^T, $(\theta(t))_{t\in T} \in \Theta \subset H^T$ is a useful signal,

$$(\Delta(t,\cdot))_{t\in T} : \Omega \to H^T$$

is any Gaussian process (so called a "white noise") defined on the probability space $(\Omega, \mathcal{F}, P)$. Let us denote by μ a probability Baire measure in H^T defined by

$$(\forall X)(X \in Ba(H^T) \to \mu(X) = P(\{\omega : (\Delta(t,\omega))_{t\in T} \in X\})),$$

where $Ba(H^T)$ denotes the Baire σ-algebra of subsets of H^T.

In information transmission theory the general decision is that the Baire measure λ, defined by the transformed noise

$$(\xi(t,\cdot))_{t\in T} : \Omega \to H^T$$

coincides with any shift $\mu_{\theta_0}(\theta_0 \in \Theta)$ of the measure μ, i.e.,

$$(\forall X)(X \in Ba(H^T) \to \lambda(X) = \mu_{\theta_0}(X)),$$

where $\mu_{\theta_0}(X) = \mu(X - \theta_0)$ for $X \in Ba(H^T)$.

A good estimation of the parameter θ_0 can be obtained by the so-called optimal estimation $\overline{\theta} : H^T \to \Theta$ which satisfies the following condition

$$(\forall\theta)(\theta \in \Theta \to \mu_\theta(\{(x(t))_{t\in T} : \overline{\theta}((x(t))_{t\in T}) = (\theta(t))_{t\in T}\}) = 1).$$

Actually, a necessary condition for existence of optimal estimation is $\Theta \cap Q(\mu) = \{\mathbf{0}\}$, where $Q(\mu)$ denotes the group of all admissible translations (in the sense of quasi-invariance) of the measure μ and $\mathbf{0}$ denotes the zero of H^T. Consequently, there arises the necessity of consideration of the group $Q(\mu)$ for investigation of the general problem of the filtration of the observed stochastic process. In this direction similar problems were discussed in the works of J. von Neumann (1935), S. Kakutani (1943), V. Sudakov (1959), J. Feldman (1966), I. Rozanov (1968), Xi Dao Xing (1972), A. Skorokhod (1975), H. Shimomura (1975), A. Kharazishvili (1985) and so on. At the present time a rich methodology of quasiinvariant measures has been created which essentially is applied for investigation of actual problems of the general theory of infinite-dimensional systems (see, e.g., [31],[64],[80],[88],[161],[178]).

For some time, an application of the methods of the theory of invariant measures in infinite-dimensional topological vector spaces was impossible because of a such an "elementary" reason that the problem of the definition of partial analogs of Lebesgue measures in such spaces was open and was connected with a number of difficulties. Hence, mathematicians were focused on the following problem:

What measures in infinite-dimensional topological vector spaces can be assumed as partial analogs of the n-dimensional classical Lebesgue measure (defined on the Euclidean vector space E_n)?

In this direction, the results of I. Girsanov and B. Mityagyn [50] and Sudakov [170] about nonexistence of nontrivial translation-invariant σ-finite Borel measures in infinite-dimensional topological vector spaces were very important. Their results asserted that the properties of σ-finiteness and of translation-invariance were not consistent. Supported by this result, one group of mathematicians was forced to weaken the property of translation-invariance for analogs of the Lebesgue measure and was trying to construct nontrivial σ -finite Borel measures which are invariant under everywhere dense linear manifolds. In this context, a result of A. Kharazishvili [87] was of special interest. He constructed a nontrivial σ-finite Borel measure in infinite-dimensional separable Hilbert space ℓ_2 which is invariant under an everywhere dense linear manifold consisting of eventually zero sequences. The second group of mathematicians has investigated the problem of the existence of nontrivial translation-invariant measures on such spaces without the property of σ-finiteness. In this direction, the results of C. Rogers [155] and D. Fremlin [45] were of interest.

The necessity of the consideration of Borel measures on infinite-dimensional topological vector spaces (which are not σ-finite) can be based on the following discussion:

Let B be an infinite-dimensional separable Banach space. Let P be any sentence formulated for elements in B and let μ be any probability Borel measure on B. Let us discuss what information the following sentence yields

"*μ-almost every element of B satisfies the property P.*"

Note here that if B is separable then an arbitrary non-zero σ-finite Borel measure defined on B is concentrated on the union of countable compact subsets $(F_k)_{k\in N}$ in B (cf.[86]) and for arbitrary $k \in N$ there exists a vector $v_k \in B$ which spans a line L_k such that every translation of L_k meets F_k in at most one point(cf.[70],p.225,Fact 8). In such a way, the support of μ may be regarded as the union of a countable family of "surfaces". Hence, the information provided by above-mentioned sentence, in general, may be very poor. For this reason to study the behavior of various general systems defined in infinite-dimensional separable Banach spaces in terms of any partial σ-finite Borel measure is not recommended and needs to extend the measure theoretic terms "measure zero" and "almost every" in terms of such Borel measures, which are not concentrated in the poor sets. In this direction, we focus on partial analogs of Lebesgue measures (which are translation-invariant and quasi-finite, but are not σ-finite) in infinite-dimensional separable Banach spaces and introduce new classes of null sets in terms of such measures in Solovay's model. Note that the support for such a measure is prevalence in the sense of [70]. The σ-ideal of null sets generated by such a measure is contained in the class of shy sets introduced by J.Christensen [29] and in general, above-mentioned information formulated in terms of such a measure is more reliable then analogous information formulated in terms of any non-zero σ-finite Borel measure. Here we demonstrate also that every shy set defined by any finite-dimensional probe is a quasi-finite Lebesgue null set, too and give the new interpretations of several interesting early obtained results in terms of partial analogs of Lebesgue measures.

Above-mentioned discussions concerned with quasiinvariant and invariant measures, show us the necessity of an investigation of their main properties. The present monograph deals with certain aspects of above-mentioned theory and some of their applications. Problems considered in this monograph are actual and considered in a large number of publications.

Here we assume that the reader is acquainted with main methods of the measure theory, of the various set-theoretical formal systems techniques and infinite combinatorics, elements of group theory and of functional analysis, which has been systematically applied in this monograph.

In 1948 Kakutani (cf.[79]) characterized a group of all admissible translations (in the sense of quasiinvariance) of the Borel probability product-measures in $\mathbf{R}^N$, where $\mathbf{R}^N$ denotes a vector space of all real-valued sequences equipped with Tykhonoff topology. This method has remained a powerful tool to characterize a group of all admissible translations of Borel probability measures with domains in various infinite-dimensional vector spaces (cf.[31],[39],[52],[165]). Similar problems are under consideration for Borel extensions of the probability product-measures in R^α (in the case of an arbitrary parameter set α). The main technique of our investigation, which applies the Kakutani method, is supported also on the specific properties of Tykhonoff topology. Using this method a general assertion is obtained, such that one result of A.Kharazishvili [88] may be obtained as its partial realization. In 1981 the French mathematician M. Talagrand [171] proved that an arbitrary cylindrical Gaussian measure μ defined in $\mathbf{R}^I$ is τ-smooth for an arbitrary parameter set I. This result immediately implies that the measure μ has a unique Borel extension in $\mathbf{R}^I$. Here we present a different proof of this result for canonical Gaussian measures and give its several interesting applications.

By using the method of separating families of real-valued functions, we present the solution of the problem of the existence of Radon quasi-invariant probability measures in the topological space $\mathbf{R}^I$ for $\mathrm{card}(I) > c$, where c denotes the cardinality of the continuum. Using the method of an isomorphic embedding we give a construction of such nontrivial σ-finite Radon measures in $\mathbf{R}^c$ which are invariant under everywhere dense vector subspaces. We give also a description of a linear manifold of all admissible (in the sense of invariance) for a sufficiently large class of σ-finite Borel measures in $\mathbf{R}^N$.

Here we also give a construction of a standard translation-invariant Borel measure in $\mathbf{R}^N$, which obtains the value one on the infinite-dimensional cube $[0,1[^N$. Actually, we are free from the demand of σ-finiteness, because the space R^N is covered by the uncountable family of pairwise disjoint shifts of $[0,1[^N$. Measures with above-mentioned properties are adopted as partial analogs of the Lebesgue measure in the infinite-dimensional topological vector space $\mathbf{R}^N$. Partial analogs of the Lebesgue measure in general Banach spaces are assumed as translation-invariant Borel measures which obtain the numerical value one on the unit sphere or on the standard infinite-dimensional parallelepiped (generated by any basis). The fundamental works of English mathematicians C. Rogers[155] and D. Fremlin [45] are devoted to problems of the existence of such measures. Here we consider the following problem posed by C. Rogers (1998):

Does there exist a such translation-invariant Borel measure in ℓ^∞ which obtains the numerical value one on the closed unite sphere?

An approximate solution of this problem is given in this monograph. In particular, applied methods of the theory of a Haar measure, a construction of a translation-invariant Borel measure defined in ℓ^∞ is given in the well-known Solovay's [167] model such that this measure takes the value 1 on the unite sphere. Note that many problems of this area of mathematics are intersected with the problems of the theory of shy-sets elaborated by B.R. Hurt, T. Sauer and J.A. Yorke in [70]. We demonstrate that in Solovay's model an arbitrary set of ν_p -measure zero is shy-set in ℓ_p, where ν_p denotes a such translation-invariant Borel measure in ℓ_p $(p \geq 1)$, which yields a numerical value one the compact set $\prod_{k\in N}[-\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}}]$. It is showed also that the opposite relation is not valid. A detailed description of some new results obtained in the theory of shy-sets is given in this monograph.

Here we give a construction of various invariant extensions of Haar measures. This construction is based on the combinatorial methods of infinite sets. This method has recently been successfully used in different areas of mathematics. Among them, special mention should be made of the method of construction of a maximal (in the sense of cardinality) family of independent sets in an arbitrary infinite basic space. The question of the existence of a maximal (in the sense of cardinality) family of independent subsets of E was considered by A. Tarski [107]. He proved that this cardinality is equal to $2^{\mathrm{card}(E)}$. This result found an interesting application in general topology by means of which it was proved that in an arbitrary infinite space the cardinality of the class of all ultrafilters is equal to $2^{2^{\mathrm{card}(E)}}$. Using the method of independent sets in the case of the Euclidean space E_n, A. Kharazishvili constructed a maximal (in the sense of cardinality) family of orthogonal elementary D_n-invariant extensions of the Lebesgue measure [85]. E. Szpilrajn (E. Marczewski)[169] was

the first who suggested the method of a construction of nonseparable extensions of the Lebesgue measure l_n. In 1950 S. Kakutani, J. Oxtoby [80], later K. Kodaira and S. Kakutani [99] constructed nonseparable invariant extensions of the Lebesgue measure. In 1974, the method of independent sets was used by A.Kharazishvili to construct an example of a nonelementary D_n-invariant extension of the Lebesgue measure μ such that the topological weight of the metric space $(\mathrm{dom}(\mu), \rho_\mu)$ associated with the measure μ was maximal (see, e.g., [84],[85]);

In this monograph, the method of independent sets of A. Tarski is generalized and successfully used to construct a maximal family of orthogonal invariant (elementary and nonelementary) extensions of the Haar measure defined on an arbitrary locally-compact σ-compact topological group. One method of the construction of nonelementary invariant extensions of the Haar measure is considered, by means of which we solve the following problem posed by A.B. Kharazishvili [85] in 1977:

Does there exist a D_n-measure μ in the Euclidean space such that some μ-measurable subset has only one point of density with respect to the Vitali standard system generated by the family of all n-dimensional cubes of the space $\mathbf{R}^n$?

In the general theory of statistical decisions there often arises a question of transition from a weakly separated family of probability measures to the corresponding strictly separated family. In 1981 A. Skorokhod [73] obtained a result which stated that if the Continuum Hypothesis is true then an arbitrary weakly separated family of probability measures, whose cardinality is not greater than the cardinality of the continuum is strictly separated. In this monograph we show that a converse relation is also valid. In particular, we demonstrate that if an arbitrary weakly separated family of probability measures whose cardinality is less or equal to the cardinality of the continuum is strictly separated, then the Continuum Hypotheses is true. Applying Martin's axiom, in 1984 Z. Zerakidze [180] proved that an arbitrary weakly separated family of Borel probability measures defined in the Polish space is strictly separated if its cardinality is not greater than the cardinality of the continuum. In this monograph, we give a generalization of this result for such complete metric spaces whose topological weight are not measurable in the wider sense.

The following aspects of measure theory are considered in this monograph:

1. The methods for the construction of some dynamical systems in infinite-dimensional vector spaces;
2. The behavior of dynamical systems under motion generated by one parameter groups of transformations of corresponding spaces and an investigation their some properties (for example, invariance, quasi-invariance, ergodicity, metrical transitivity, stability in the sense of Poisson, topological stability, stability in the sense of Steinhauss, uniqueness property, analogue of 0-1 law, strict transitivity property, separation property, essential uniqueness, etc);
3. An analog of Lioville's theorem in infinite-dimensional vector space $\mathbf{R}^{\mathbf{N}}$ and a description of the class of all continuous asymmetrical unit impulses under the action of which the designated phase flows preserve the conservation property;

4. A discovering of the deep connections between dynamical systems and various formal systems of mathematical logic and set theory (for example, Zermelo-Fraenkel's model, Martin's model, Solovay's model, Mycielski-Steinhaus model, etc.);

5. Some applications of the techniques of infinite combinatorics (independent families of sets, almost disjoint families of sets, separating families of functions, etc.) to measure extension problems and the method of construction of various (elementary, nonelementary, separable, nonseparable) invariant extensions of the Haar measure;

6. An investigation of the property of quasi-invariance of the entered and transformed signals ("white noise", Gaussian process) in terms of Gaussian measures on infinite-dimensional vector spaces.

7. An elaboration heuristic algorithm for the "Taxonomy" problem to recognize forms in the Martin-Solovay's model and some of its applications in information theory to recognize the form of entered and transformed white noise.

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Chapter 1

Basic Concepts

In this preliminary chapter we introduce and recall some notations and elementary facts from set theory, general topology and measure theory. We will systematically use these facts in our further considerations.

The symbol ZF denotes the so-called Zermelo-Fraenkel set theory which is one of the most important formal systems of axioms in modern set theory (cf. [12]). The basic notions of the Zermelo-Fraenkel system are sets and the membership relation $\in$ between them. The ZF system consists of several axioms which formalize various properties of sets in terms of the relation $\in$. We are not going to point out here the precise list of these axioms and will actually work within the framework of the so-called "naive set theory".

The symbol ZFC denotes the Zermelo-Fraenkel theory with the Axiom of Choice AC. In other words, ZFC is the theory

$$ZF\&AC,$$

where AC stands for the Axiom of Choice.

Presently, one is well aware of the fact that the ZFC theory forms the basis of all modern mathematics, i.e., almost all modern mathematical areas can be developed starting with the ZFC theory. The Axiom of Choice is a very powerful set-theoretical assertion which gives rise to many unusual and interesting consequences. Sometimes, in order to get some required result, we do not require the whole power of the Axiom of Choice. In such cases, it is sufficient to apply one of its weak forms.

If x and X are any two sets, then the relation $x \in X$ means that x belongs to X. In this situation we also say that x is an element of X.

The relation $X \subseteq Y$ means that a set X is a subset of a set Y.

The relation $X \subset Y$ means that a set X is a proper subset of a set Y.

If $\mathbf{R}(x)$ is a relation depending on an element x (or, in other words, $\mathbf{R}(x)$ is a property of an element x), then the symbol

$$\{x : \mathbf{R}(x)\}$$

denotes the set (the family, the class) of all those elements x for which the relation $\mathbf{R}(x)$ holds.

The symbol $\emptyset$ denotes as usual the empty set, i.e.,

$$\emptyset = \{x : x \neq x\}.$$

If X is any set, then the symbol $P(X)$ denotes the family of all subsets of X, i.e., we have

$$P(X) = \{Y : Y \subseteq X\}.$$

The set $P(X)$ is also called the power set of a given set X.

Let X and Y be any two sets. Then as usual:

$X \cup Y$ denotes the union of X and Y;

$X \cap Y$ denotes the intersection of X and Y;

$X \setminus Y$ denotes the difference of X and Y;

$X \triangle Y$ denotes the symmetric difference of X and Y, i.e.,

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X).$$

Let X be an arbitrary nonempty set and $\Re$ be some class of its subsets. The class $\Re$ is called a ring of subsets of X if the following condition holds:

$$(\forall X_1)(\forall X_2)(X_1 \in \Re \ \& \ X_2 \in \Re \rightarrow (X_1 \cup X_2 \in \Re) \ \& \ (X_1 \cap X_2 \in \Re) \ \&(X_1 \setminus X_2 \in \Re)).$$

If the condition $X \in \Re$ holds too, then a ring $\Re$ is called an algebra of subsets of X.

A ring (an algebra) $\Re$ is called a σ-ring (σ-algebra) of subsets of X if

$$(\forall k)(\forall X_k)(k \in N \ \& \ X_k \in \Re \rightarrow \cup_{k \in N} X_k \in \Re).$$

A measurable space is a pair (E,S), where E is a nonempty set and S is an σ-algebra of subsets of E. Each element X of S is called a measurable subset of E.

We place

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

The set $X \times Y$ is called a Cartesian product of given sets X and Y.

In a similar way, applying recursion, one can define the Cartesian product

$$X_1 \times X_2 \times \cdots \times X_n$$

of a finite family $\{X_1, X_2, \cdots, X_n\}$ of arbitrary sets.

If X is a set, then the symbol $\mathrm{card}(X)$ denotes the cardinality of X. Sometimes $\mathrm{card}(X)$ is also called the cardinal number of X.

ω is the first infinite cardinal (ordinal) number. In fact, ω is the cardinality of the set

$$N = \{0, 1, 2, \cdots, n, \cdots\}$$

of all natural numbers. Sometimes it is convenient to identify the sets ω and N.

ω_1 is the first uncountable cardinal (ordinal) number. Notice that, as usual, ω_1 is identified with the set of all countable ordinal numbers (countable ordinals).

Various ordinal numbers (ordinals) are denoted by

$$\alpha, \beta, \gamma, \xi, \cdots.$$

Let α be an ordinal number. We say that α is a limit ordinal if

$$\alpha = \sup\{\beta : \beta < \alpha\}.$$

The cofinality of a limit ordinal α is the smallest ordinal ξ such that there exists a family

$$\{\alpha_\iota : \iota < \xi\}$$

of ordinals satisfying the relation

$$\alpha_\iota < \alpha \ \ (\iota < \xi),$$

$$\alpha = \sup\{\alpha_\iota : \iota < \xi\}.$$

The cofinality of a limit ordinal α is denoted by the symbol $cf(\alpha)$.

Clearly, we have the inequality

$$cf(\alpha) \leq \alpha$$

for all limit ordinal numbers α.

A limit ordinal number α is called a regular ordinal if

$$cf(\alpha) = \alpha.$$

A limit ordinal number α is called a singular ordinal if

$$cf(\alpha) < \alpha.$$

For example, ω and ω_1 are regular ordinals (cardinals) and ω_ω is a singular ordinal (cardinal).

If k is an arbitrary infinite cardinal number, then the symbol k^+ denotes the smallest cardinal among all those cardinals which are strictly greater than k. For example, we have

$$\omega^+ = \omega^1, \omega_2 = (\omega_1)^+, \cdots.$$

The symbol Q denotes the set of all rational numbers.

The symbol $\mathbf{R}$ denotes the set of all real numbers. If the set $\mathbf{R}$ is equipped with the standard mathematical structures (order structure, algebraic structure, topological structure), then $\mathbf{R}$ is usually called the real line.

Let n be a fixed natural number. The symbol $\mathbf{R}^n$ (respectively, S^n) denotes as usual the n-dimensional Euclidean space (respectively, the n-dimensional Euclidean unit sphere).

Let X and Y be two sets. A binary relation between X and Y is an arbitrary subset G of the Cartesian product of X and Y, i.e.,

$$G \subseteq X \times Y.$$

In particular, if $X = Y$, then we say that G is a binary relation on the basic set X.

For a binary relation $G \subseteq X \times Y$, we put

$$\mathrm{pr}_1(G) = \{x : (\exists y)((x,y) \in G)\}, \ \mathrm{pr}_2(G) = \{y : (\exists x)((x,y) \in G)\}.$$

It is clear that

$$G \subseteq \mathrm{pr}_1(G) \times \mathrm{pr}_2(G).$$

The Axiom of Dependent Choices is the following set-theoretical statement:

If G is a binary relation on a nonempty set X and, for each element x of X, there exists an element y of X such that $(x,y) \in G$, then there exists a sequence $(x_0, x_1, \cdots, x_n, \cdots)$ of elements of X such that $(x_n, x_{n+1}) \in G$ for all $n \in N$.

The Axiom of Dependent Choices is usually denoted by *DC*. Actually, the statement *DC* is a weak form of the Axiom of Choice which is completely sufficient for most areas of classical mathematics: geometry of a finite-dimensional Euclidean space, mathematical analysis of the real line, the Lebesgue measure theory and so on.

We will consider some important results and facts from measure theory which have various interesting applications to the theory of invariant and quasiinvariant measures in infinite-dimensional vector spaces.

The following notion of a specific kind of partial ordering frequently turns out useful in studying various questions of measure theory and general topology.

A partial ordering $(T, \preceq)$ is called a tree if T has the least element and, for each $y \in T$ the set $\{x \in T : x \preceq y\}$ is well-ordered by $\preceq$. The least element of T is called a root of T. For any ordinal number α the α-th level of T is the set

$$T_\alpha = \{y : \{x \in T : x \prec y\} \text{ has order type } \alpha\}.$$

The height of a tree is a least ordinal α such that the α-th level of T is empty.

Let A be a non-empty set and let α be an ordinal. A complete A-ary tree of height α, which consists of all functions from $\cup_{\beta \prec \alpha} A^\beta$ and is ordered by inclusion, is denoted by $A^{\prec \alpha}$. If $A = \{0; 1\}$, then complete A-ary trees are called binary trees.

Any linearly ordered subset of the tree $(T, \preceq)$ is called a branch in T. A subset P of T is called a path through the tree T if P is a branch and contains exactly one element from each nonempty level of T.

The following fundamental statement was established by König (cf.[25]).

Theorem 1.1 *Suppose that $(T, \preceq)$ is a tree of height ω such that all levels of T are finite. Then there exists a path through T.*

Proof. Let x_0 be the root of T. For each $n \in \omega \setminus \{0\}$, we can recursively pick an element $x_n \in T_n$ such that $x_n \succ x_{n-1}$ and the set

$$\{y \in T : x_n \prec y\}$$

is infinite. This is possible since every level T_n of T is finite. Then $(x_n)_{n \in N}$ is a path through T. $\square$

Let us recall some notions which are necessary to formulate the Axiom of Determinacy.

An arbitrary subset $A \subset \omega^\omega$ determines a game of type G_A between two players denoted by I and II. A game of type G_A is described as follows:

Player I writes a natural number a_0. His opponent player II, knowing the number a_0, writes a number a_1. Player I, knowing the number a_1 and remembing his number a_0, writes

his new number a_2. Further, player II, looking at a_0, a_1, a_2, writes a number a_3, and so on. In this infinite game, the sequence of natural numbers $\alpha = (a_0, a_1, \cdots,)$ is obtained. If this summarizing sequence belongs to the set A, then player I wins a game of type G_A. Otherwise, player II wins this game in other cases.

Various kind of games (for example, chess, checkers and so on) can be represented by the above scheme.

Let $\omega^{(\omega)}$ be the set of all finite sequences of natural numbers, including an empty set. The last set can be considered as a sequence of length zero.

A function $\sigma : \omega^{(\omega)} \to \omega$ is called a strategy in the game of type G_A.

Let σ be a strategy for the player I and τ be a strategy for the player II.

According to the strategies σ and τ, we have

$$a_0 = \sigma(\emptyset),$$

$$a_1 = \tau(a_0) = \tau(\sigma(\emptyset)),$$

$$a_2 = \sigma(a_0, a_1) = \sigma(\sigma(\emptyset), \tau(\sigma(\emptyset))), \cdots.$$

A function $\sigma : \omega^{(\omega)} \to \omega$ is called a winning strategy for the player I if

$$(a_0, a_1, a_2, \cdots) \in A$$

for an arbitrary strategy τ for the player II.

Analogously, a function $\tau : \omega^{(\omega)} \to \omega$ is called a winning strategy for the player II if

$$(a_0, a_1, a_2, \cdots) \notin A$$

for an arbitrary strategy σ for the player I.

An infinite game of type G_A is called determined if there exists a winning strategy for at least one of two players in this game.

The Axiom of Determinacy (denoted by AD) is the assertion that every infinite game of type G_A (for arbitrary $A \subseteq \omega^\omega$) is determined.

The classical theorem of Mycielski and Swierczkowski is formulated as follows.

Theorem 1.2 *If the Axiom of Determinacy is valid, then every subset of the real axis* **R** *is Lebesgue measurable.*

The proof of Theorem 1.2 can be found in [81].

The following result is a simple consequence of Theorem 1.2.

Theorem 1.3 *The Axiom of Determinacy contradicts the Axiom of Choice.*

The symbol **c** denotes the cardinality of the continuum, i.e.,

$$\mathrm{c} = 2^\omega = card(R).$$

The Continuum Hypothesis (shortly, CH) is the assertion that

$$\mathrm{c} = \omega_1.$$

K. Gödel showed that the (ZFC) & (CH) theory is consistent since it is valid in the Constructible Universe (see [77]). On the other hand, P. Cohen showed that the

$$(ZFC)\&(\neg CH)$$

theory is also consistent (see [26],[27]). Hence the sentence CH is independent of the ZFC theory.

A stronger form of the Continuum Hypothesis is the Generalized Continuum Hypothesis, denoted by GCH, which states that

$$(\forall k > \omega)(2^k = k^+).$$

K. Gödel also showed that the (ZFC) & (GCH) theory is consistent because it is valid in the Constructible Universe [53].

From the moment it turned out that the Continuum Hypothesis was independent of ZFC, the issue of adding new axioms became really unbalanced. While the Continuum Hypothesis is an extremely powerful assertion, perhaps even too strong, its negation is rather weak. Hence a natural necessity arose to find an appropriate axiom which even in the absence of the Continuum Hypothesis could give tools efficient enough for mathematical constructions. Martin's axiom turned out to be a very good candidate to fill this place.

To formulate Martin's Axiom, we need some notations and definitions.

Let P be an arbitrary nonempty set.

A binary relation of $G \subseteq P \times P$ is called an equivalence relation on P if the following three conditions hold:

1) $(p,p) \in G$ for all elements $p \in P$;
2) $(p,q) \in G$ and $(q,r) \in G$ imply $(p,r) \in G$;
3) $(p,q) \in G$ implies $(q,p) \in G$.

If G is an equivalence relation on P, then the pair (P,G) is called a set equipped with an equivalence relation.

Obviously, if G is an equivalence relation on P, then we have a partition of P canonically associated with G. This partition consists of the sets

$$G(p) \ \ (p \in P),$$

where $G(p)$ denotes the section of G corresponding to an element $p \in P$; in other words,

$$G(p) = \{q : (p,q) \in G\}.$$

Conversely, every partition of the set P canonically defines an equivalence relation on P.

Let P be an arbitrary set and let G be a binary relation on P. We say that G is a partial order on P if the following three conditions hold:

1) $(p,p) \in G$ for each element p of P;
2) $(p,q) \in G$ and $(q,r) \in G$ imply $(p,r) \in G$;
3) $(p,q) \in G$ and $(q,p) \in G$ imply $p = q$.

Suppose that G is a partial order on a set P. As usual, we write

$$p \preceq q \ \text{ iff } \ (p,q) \in G.$$

The pair $(P,\preceq)$ is called a partially ordered set.

Let $(P,\preceq)$ be a partially ordered set. We say that a set $D \subseteq P$ is dense in P if for each $p \in P$ there is $q \in D$ with $p \preceq q$. (This is actually the density in the topological sense for a suitable topology on P, viz., that having as a base all sets of the form $\{p : p \succeq q\}$, where $q \in P$).

A set $D \subseteq P$ is called a subnet if for arbitrary two elements $p_1 \in P$ and $p_2 \in P$ there exists an element $q \in D$ such that $p_1 \preceq q$ and $p_2 \preceq q$. Two elements p and q are called compatible if there is an element $r \in P$ such that $p \preceq r$ and $q \preceq r$. Finally, we say that $(P,\preceq)$ satisfies the countable chain condition (or, simply, c.c.c.) if every uncountable subset of P contains at least two compatible elements.

Martin's Axiom usually denoted by MA is the following statement:

If $(P,\preceq)$ is a partially ordered set satisfying c.c.c. and $(D_i)_{i\in I}$ is a family of dense subsets in P with $card(I) < 2^{\omega}$, then there is a subnet $D \subseteq P$ which intersects each D_i.

The Continuum Hypothesis readily implies Martin's Axiom. Indeed, let us assume CH. Let $(P,\preceq)$ be any partially ordered set and let $(D_n)_{n\in\omega}$ be any sequence of dense subsets of P. Then we can recursively construct an increasing sequence $(p_n)_{n\in\omega}$ of elements of P such that $p_n \in D_n$ for each $n \in \omega$. Now, put

$$D = \{p \in P : (\exists n \in \omega)(p_n \preceq p)\}.$$

Evidently, D is a subnet in P which intersects every D_n.

Martin and Solovay proved that the statement $(MA)\&(\neg CH)$ is consistent with ZFC (see, e.g. [77]).

Let E be a basic set (in general, E is assumed to be infinite) and let T be a topological structure on E, i.e., a topology on E. As a rule, the pair (E,T) is called a topological space.

Let (E,T) be an arbitrary topological space.

We say that a set $X \subseteq E$ is nowhere dense (in E) if $\mathrm{int}(\mathrm{cl}(X)) = \emptyset$.

We say that a set $Y \subseteq E$ is a first category subset of E if Y can be represented in the form $Y = \cup_{n\in\omega} Y_n$, where all $Y_n (n \in \omega)$ are nowhere dense subsets of E.

We say that a set $Z \subseteq E$ is a second category subset of E if Z is not a first category subset of E.

Let us consider some facts from general topology which we will need to make use of in the sequel. It is understood that the reader knows some elementary facts and notions from this area of mathematics, for instance, such notions as continuous mapping, separation axioms and their corresponding classes of topological spaces (Hausdorff spaces, regular spaces, completely regular spaces, normal spaces and other spaces), quasi-compactness, metrizability of a topological space, completeness for metric spaces and so on (see, e.g., [34] or [83]).

The notion of quasi-compactness is one of the most important ones among the topological notions listed above. Let us recall that a topological space E is quasi-compact if every open covering of E contains a finite subcovering of E.

A space E is called compact if it is Hausdorff[1] and quasi-compact at the same time. There are a lot of remarkable theorems connected with the notion of quasi-compactness. The main one, of course, is the classical Tykhonoff theorem.

[1] A topological space G is called Hausdorff if for arbitrary two different points $x, y \in G$ there exists open sets G_x and G_y such that $x \in G_x$, $y \in G_y$ and $G_x \cap G_y = \emptyset$.

Theorem 1.4 (Tykhonoff) *The topological product of an arbitrary family of quasi-compact spaces is a quasi-compact space. Conversely, if the topological product of the family of nonempty spaces is quasi-compact, then each of these spaces is quasi-compact.*

The proof of this theorem can be found in [34] or [82].

Theorem 1.5 (Banach) *Let E be any topological space and let $(V_i)_{i\in I}$ be any family of open first category subsets of E. Then the union $V=\cup_{i\in I}V_i$ is an open first category set, too.*

The proof of this theorem is given in [25].

From Theorem 1.5 we immediately obtain the following result.

Theorem 1.6 *Any topological space E can be represented as the union*

$$E=E_1\cup E_2,$$

where the set E_1 is an open first category subset of E, the set E_2 is a closed Baire subspace of E and $E_1\cap E_2=\emptyset$. Similarly, any topological space E can be represented as the union

$$E=E_1^*\cup E_2^*,$$

where the set E_1^* is a closed first category subspace of E, the set E_2^* is an open Baire subspace of E and $E_1^*\cap E_2^*=\emptyset$.

The following classical definition plays one of the main roles in this work.

Let E be any topological space and let X be a subset of E. We say that the set X has the Baire property in E if X can be represented as

$$X=(V\cup Y)\setminus Z,$$

where V is an open subset of E, and Y and Z are first category subsets of E. It is easy to see that a set $X\subseteq E$ has the Baire property if and only if there exist an open set $V\subseteq E$ and a first category set $P\subseteq E$ such that

$$X=V\triangle P.$$

The class of all sets which have the Baire property in E will be denoted by the symbol $\overline{B}(E)$ and the class of all first category subsets of E- by the symbol $K(E)$.

It is easy to prove the following theorem.

Theorem 1.7 *The class $\overline{B}(E)$ is a σ-algebra of subsets of E. This σ-algebra is generated by the union $T(E)\cup K(E)$, where $T(E)$ is the topology of the space E.*

Let us recall that the Borel σ-algebra of a topological space E is the σ-algebra generated by the topology $T(E)$. The Borel σ-algebra of the topological space E is denoted by the symbol $B(E)$. Every element $X\in B(E)$ is called a Borel set in E.

It is clear that the inclusion

$$B(E)\subseteq\overline{B}(E)$$

always holds true.

Let (E_1,S_1) and (E_2,S_2) be two measurable spaces and let f be a mapping from E_1 into E_2. We say that f is a measurable mapping if

$$(\forall X)(X\in S_2\rightarrow f^{-1}(X)\in S_1).$$

We say that a mapping $f : E_1 \to E_2$ is a measurable isomorphism from E_1 onto E_2 if f is a measurable bijection and the inverse mapping f^{-1} is measurable, too.

Let E_1 and E_2 be two topological spaces. We can consider two measurable spaces $(E_1, B(E_1))$ and $(E_2, B(E_2))$. Let f be a mapping from E_1 into E_2. Let us recall that f is called a Borel mapping if f is a measurable mapping from E_1 into E_2.

We say that a mapping

$$f : E_1 \to E_2$$

has the Baire property if, for each open set $V \subseteq E_2$, the set $f^{-1}(V)$ has the Baire property in the space E_1.

It is easy to show that the following statement is true.

Theorem 1.8 *If E_1 and E_2 are topological spaces and if f is a mapping from E_1 into E_2, then the next three statements are equivalent:*

1) the mapping f has the Baire property;

2) for any closed set $Z \subseteq E_2$ the set $f^{-1}(Z)$ has the Baire property in the space E_1;

3) for any Borel set $Z \subseteq E_2$ the set $f^{-1}(Z)$ has the Baire property in the space E_1.

In particular, any Borel (hence, any continuous) mapping $f : E_1 \to E_2$ has the Baire property.

The following useful result is valid.

Theorem 1.9 *Let E_1 and E_2 be two topological spaces such that E_2 satisfies the second countability axiom (i.e., there exists a countable base for $T(E_2)$). Let f be a mapping from E_1 into E_2. Then the next two statements are equivalent:*

1) the mapping f has the Baire property;

2) there exists a first category set $Z \subseteq E_1$ such that the restriction of the mapping f to the set $E_1 \setminus Z$ is continuous.

The proof of Theorem 1.9 can be found in [25].

Now we introduce some simple cardinal-valued functions describing various properties of topological spaces. We put

$$\omega(E) = \inf\{\mathrm{card}(B) : B \text{ is a base of } T(E)\} + \omega;$$

$$\mathrm{c}(E) = \sup\{\mathrm{card}(B) : B \text{ is a family of pairwise disjoint nonempty open sets}$$

$$\text{in } E\} + \omega;$$

$$d(E) = \inf\{\mathrm{card}(X) : X \text{ is a dense subset of } E\} + \omega;$$

$$\pi\omega(E) = \inf\{\mathrm{card}(B) : B \subseteq T(E) \setminus \{\emptyset\} \text{ and } B \text{ is coinitial in } (T(E) \setminus \{\emptyset\}, \subseteq)\} + \omega.$$

These functions are usually referred to as:

$\omega(E)$ is the weight of a space E;

$c(E)$ is the Suslin number of a space E;

$d(E)$ is the density of a space E;

$\pi\omega(E)$ is the π-weight of a space E.

A topological space E is said to satisfy the Suslin condition if $c(E) = \omega$.

A topological space E is called separable if $d(E) = \omega$.

A family $B \subseteq T(E) \setminus \{\emptyset\}$ is said to be a π-base of the topological space E if B is a coinitial subset of $(T(E) \setminus \{\emptyset\}, \subseteq)$

The following inequalities are obvious:

$$c(E) \leq d(E) \leq \pi\omega(E) \leq \omega(E).$$

As is well known, for a metric space E all these inequalities become equalities (see, e.g., [25]).

It is also easy to prove that the inequality

$$card(E) \leq 2^{2^{d(E)}}$$

holds for any Hausdorff topological space E.

Example 1.1 Let α be any infinite cardinal. Let us consider the space $\mathbf{R}^{\alpha}$ equipped with the product topology. It can be shown that this space satisfies the Suslin condition, i.e., $c(\mathbf{R}^{\alpha}) = \omega$. Moreover , if $\alpha = 2^{\omega}$, then it can be shown that the space $\mathbf{R}^{\alpha}$ is separable, i.e., $d(\mathbf{R}^{\alpha}) = \omega$. Indeed, in that case the space $\mathbf{R}^{\alpha}$ can be identified with the space $\mathbf{R}^{[0,1]}$ consisting of all real functions defined on $[0,1]$, equipped with pointwise convergence topology. Using, for instance, the classical Weierstrass theorem on approximation, we deduce that the countable set of all polynomials on $[0,1]$ with rational coefficients is dense in the space $\mathbf{R}^{[0,1]}$.

If $\alpha > 2^{\omega}$, then the topological space $\mathbf{R}^{\alpha}$ is not separable but, as mentioned above, it always satisfies the Suslin condition.

Next, we will introduce one notion which is important for our purposes and plays a remarkable role in classical descriptive set theory.

A topological space E is a Polish space if E is homeomorphic to a complete separable metric space.

It is clear that any compact metric space is a Polish space. In particular, the Cantor discontinuum $\{0,1\}^{\omega}$ (where the set $\{0,1\}$ is equipped with the discrete topology) is a Polish space. The space $\omega^{\omega} = N^{\omega}$, where N is equipped with the discrete topology, is another standard example of a Polish space. The space N^{ω} is usually called the canonical Baire space. It is not difficult to prove that N^{ω} is homeomorphic to the set of all irrational numbers in $\mathbf{R}$.

The following statement contains topological characterizations of some important metric spaces.

Theorem 1.10 *The following three statements hold:*

1) any nonempty Polish space is a continuous image of the space N^{ω}*;*

2) any nonempty compact metric space is a continuous image of the Cantor discontinuum $\{0,1\}^{\omega}$*;*

3) any separable metric space is topologically contained in the Hilbert cube $[0,1]^{\omega}$.

The proof of Theorem 1.10 can be found in [25].

In context of Theorem 1.10 the following proposition is of some interest (cf. [94]).

Theorem 1.11 *An arbitrary infinite-dimensional separable Banach space H is the continuous linear image of the space ℓ_1, where*

$$\ell_1 = \{(x_k)_{k\in N} \in \mathbf{R}^N \ \& \ \sum_{k\in N} |x_k| < \infty\}.$$

Let us consider some facts from measure theory.

A measure of a set is a generalization of the notion of length of an interval, area of a plane figure, and volume of a three-dimensional body. The notion of a measurable set appeared in real function theory during investigations and generalizations of the concept of an integral. A standard example of a measure is the Lebesgue measure on the real line $\mathbf{R}$ defined by Lebesgue in 1902. It extends the notion of interval length to a much wider class of subsets of $\mathbf{R}$. This class of sets contains all Borel and all analytic subsets of the real line and many other subsets of $\mathbf{R}$ (see e.g.[112]).

Let E be a nonempty basic set, let S be any algebra of subsets of E, and let μ be a function from S into the extended real line

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$$

such that

$$\mathrm{card}(\mathrm{ran}(\mu) \cap \{-\infty, +\infty\}) \leq 1.$$ [2]

We say that μ is finitely additive (or, simply, additive) if for every finite family $\{X_1, \cdots, X_n\} \subseteq S$ of pairwise disjoint sets, we have

$$\mu(\cup_{i=1}^n X_i) = \sum_{i=1}^n \mu(X_i).$$

Similarly, we say that μ is countably additive (or σ-additive) if for every countable family $\{X_i : i \in I\} \subseteq S$ of pairwise disjoint sets such that $\cup_{i\in I} X_i \in S$ we have

$$\mu(\cup_{i\in I} X_i) = \sum_{i=1}^\infty \mu(X_i).$$

Finally, we say that a function $\mu : S \to \mathbf{R}^+$ is a measure (defined on the algebra S) if the following conditions hold:

a) $\mu(\emptyset) = 0$;

b)$(\forall X)(X \in S \to \mu(X) \geq 0)$;

c) μ is σ-additive.

A measure space is a triple (E, S, μ), where E is a nonempty basic set, S is an σ-algebra of subsets of E and μ is a measure on S.

A measure μ is called σ-finite if there exists a countable family $\{X_n : n \in N\}$ of subsets of S such that

$$\cup_{n\in N} X_n = E, \ (\forall n \in N)(\mu(X_n) < +\infty).$$

A measure μ is called finite if

$$(\forall X)(X \in S \to \mu(X) < +\infty).$$

[2] $\mathrm{ran}(\mu) = \{\mu(X) : X \in S\}$.

We say that a measure μ is a probability measure if $\mu(E) = 1$.

We say that a measure μ is complete if for every $X \in S$ with $\mu(X) = 0$ we have

$$(\forall Y)(Y \subseteq X \rightarrow Y \in S).$$

A measure μ is called nonzero (nontrivial) if $\mu(E) \neq 0$.

A measure μ is called nonatomic if

$$(\forall X)(X \in S \ \&\ \mu(X) > 0 \rightarrow (\exists Y)(Y \subset X \ \&\ Y \in S \ \&\ 0 < \mu(Y) < \mu(X))).$$

A measure μ is called diffused (or continuous) if

$$(\forall x)(x \in E \rightarrow (\{x\} \in S \ \&\ \mu(\{x\}) = 0).$$

For any nonzero measure μ we denote

$$L(\mu) = \{X : (\exists Y)(X \subseteq Y \ \&\ Y \in S \ \&\ \mu(Y) = 0)\}.$$

If S is a σ-algebra, then it is clear that the class $L(\mu)$ is a σ-ideal [3] of subsets of E. Moreover, one can see that the measure μ is complete if and only if $L(\mu) \subseteq S$.

The members of the class $L(\mu)$ are called μ-measure zero sets or μ-negligible sets.

For every measure ν, there exists the smallest (with respect to inclusion) complete measure μ extending ν. The measure μ is called a completion of the original measure μ.

The following fact is fundamental for the whole measure theory.

Theorem 1.12 (Carathéodory) *Let μ be a measure on an algebra S of subsets of the basic set E. Then there exists a measure extending the original measure μ onto the σ-algebra generated by the algebra S. If the original measure μ is σ-finite, then this extension is unique.*

The proof of Theorem 1.12 is presented e.g. in [11], [37], [54].

Suppose that μ is a measure on the algebra S of subsets of E. We define a real-valued function μ^* on the class $P(E)$ by the formula

$$\mu^* = \inf\{\sum_{n \in N} \mu(Y_n) : \{Y_n : n \in N\} \subseteq S \ \&\ X \subseteq \cup_{n \in N} Y_n\}.$$

This function is called a outer measure associated with μ.

We say that a subset Z of E is μ^*-measurable if, for any set $X \subseteq E$, the following Carathéodory condition holds:

$$\mu^*(X \cap Z) + \mu^*(X \cap (E \setminus Z)) = \mu^*(X).$$

It can be shown that the class of all μ^*-measurable sets $Z \subseteq E$ is a σ-algebra of subsets of E which contains the original algebra S. Moreover, the function μ^* considered only on the class of all μ^*-measurable sets is countably additive and therefore it is a measure. It also

[3]The class $\mathcal{T}$ of subsets of E is called σ-ideal if the following conditions hold:
$a)$ $E \notin \mathcal{T}$;
$b)$ $(\forall X)(\forall Y)(X \in \mathcal{T} \ \&\ Y \subset X \ \rightarrow Y \in \mathcal{T})$;
$c)$ if $X_k \in \mathcal{T}$ for $k \in N$ then $\cup_{k \in N} X_k \in \mathcal{T}$.

extends the original measure μ and is complete. These facts immediately imply Theorem 1.12. It is sometimes useful to consider, along with outer measures, their dual object, the so-called inner measure. Recall that if the original measure μ is defined on a σ-algebra, then a function μ_* defined on the class $P(E)$ by the formula

$$\mu_*(X) = \sup\{\mu(Y) : Y \in S \ \& \ Y \subseteq X\}$$

is called a inner measure associated with μ.

A set $X \subseteq E$ is called μ-massive (or a set with a full outer measure with respect to μ) if the equality

$$\mu_*(E \setminus X) = 0$$

holds. It is easy to see that when we have a finite measure μ, a set $X \subseteq E$ is μ-massive if and only if

$$\mu^*(X) = \mu(E).$$

The complements of μ-massive subsets of E can be successfully applied to various problems connected with measure extensions.

The following statement is valid.

Theorem 1.13 *Let (E,S,μ) be a measure space and let I be a σ-ideal of subsets of E such that*

$$(\forall Y)(Y \in I \rightarrow \mu_*(Y) = 0).$$

Then the formula

$$\nu(X \triangle Y) = \mu(X) \ (X \in S, Y \in I)$$

correctly defines a measure ν on the σ-algebra $S(I)$ which extends the original measure μ, where $S(I)$ denotes, as usual, the σ-algebra generated by the class $S \cup I$.

The proof of Theorem 1.13 is given e.g. in [25], [85].

Let E be again a basic set and let S be a σ-algebra of subsets of E. The pair (E,S) is usually called a measurable space.

A real function f on E is called S-measurable if, for every Borel set $X \subseteq \overline{\mathbf{R}}$, the set $f^{-1}(X)$ is in S. The simplest examples of S-measurable functions are characteristic functions I_Y defined by

$$I_Y(x) = \begin{cases} 1, & \text{if } x \in Y; \\ 0, & \text{if } x \notin Y, \end{cases}$$

where $Y \in S$.

Any linear combination of several characteristic functions is called a step function.

The following theorem shows that the class of all S-measurable functions is closed under all natural algebraic operations.

Theorem 1.14 *Let E be a basic set and let S be some σ-algebra of E. Suppose that*

$$\Phi : \overline{\mathbf{R}} \times \overline{\mathbf{R}} \rightarrow \overline{\mathbf{R}}$$

is a $B(\overline{\mathbf{R}} \times \overline{\mathbf{R}})$-measurable function, and suppose that f and g are two S-measurable functions. Then the function $h : E \rightarrow \overline{\mathbf{R}}$ defined by the formula

$$h(x) = \Phi(f(x), g(x)) \ (x \in E)$$

is S-measurable, too.

The proof of Theorem 1.14 can be found e.g. in [25].

Let $f : E \to \overline{R}$ be a function. We put

$$f^{+}(x) = \max\{f(x), 0\} \ (x \in E),$$

$$f^{-}(x) = \max\{-f(x), 0\} \ (x \in E).$$

It is evident that

$$0 \leq f^{+}, \ 0 \leq f^{-}, \ f = f^{+} - f^{-}.$$

From Theorem 1.14 it follows at once that the function f is S-measurable if and only if both functions f^{+} and f^{-} are S-measurable.

The class of S-measurable real functions is also closed under limit operations: if $(f_n)_{n \in N}$ is any sequence of S-measurable functions and

$$f = \lim_{n} f_n, \ g = \inf_{n} f_n, \ h = \sup_{n} f_n,$$

then f, g and h are also S-measurable functions.

Let (E, S, μ) be a measure space, let X be a μ-measurable subset of E, and let $f : E \to \mathbf{R}$ be an S -measurable and nonnegative function.

The μ-integral of the function f on the set X is defined by the formula

$$\int_X f d\mu = \sup\{\sum_{n} \inf(f|X_n) \times \mu(X_n) : \{X_n : n \in N\} \subseteq S \text{ is a partition of the set } X\}.$$

In many cases, the real number $\int\limits_X f d\mu$ is also denoted by the symbol

$$\int_X f(x) d\mu(x).$$

Suppose now that $f : E \to R$ is any S-measurable function. Then we put

$$\int_X f d\mu = \int_X f^{+} d\mu - \int_X f^{-} d\mu$$

if at least one of the integrals from the right-hand side of this equality is finite. If both integrals from the right-hand side of this equality are finite, then we say that the function f is integrable on the set X, and the real number $\int_X f d\mu$ is called a μ-integral of f on the set X. We say that the function f is μ-integrable if it is μ-integrable on the whole basic set E.

The class of all μ-integrable functions on E is a Banach space with respect to the norm

$$||f|| = \int_E (f^{+} + f^{-}) d\mu.$$

Of course, we identify here the functions which are equivalent with respect to the measure μ, i.e., we identify the functions which coincide almost everywhere (with respect to μ) on the basic set E.

We expect the reader to know some standard facts about integrable real functions such as the Lebesgue theorem on majorated convergence, the Fatou lemma, absolute continuity of integrals, etc.

Let us take one more look at the notion of a measure space with a σ-finite measure. Suppose that (E,S,μ) is such a space and assume that $\mu(E)=\infty$. Let $\{X_n : n\in N\}\subseteq S$ be a countable family of pairwise disjoint sets, such that $\cup_{n\in N}X_n = E$ and

$$0<\mu(X_n)<+\infty$$

for each $n\in N$. Let us consider the measure ν on the σ-algebra S defined by the formula

$$\nu(X)=\sum_{n\in N}\frac{1}{2^{n+1}}\frac{\mu(X\cap X_n)}{\mu(X_n)}\ (X\in S).$$

Observe that ν is a probability measure on S. If X is an arbitrary set from S, then $\nu(X)>0$ if and only if $\mu(X)>0$. In this case we say that the measures μ and ν are equivalent.

Let I be a nonempty set of indices and suppose that $((E_i,S_i,\mu_i))_{i\in I}$ is a family of probability spaces. In the usual way, we define the product

$$(E,S,\mu)=\prod_{i\in I}(E_i,S_i,\mu_i)$$

of this family of measure spaces.

Let $E=\prod\limits_{i\in I}E_i$ be the Cartesian product of the family of basic sets $(E_i)_{i\in I}$. A subset X of E is called a rectangle if it can be represented in the form

$$X=\prod_{i\in I}X_i,$$

where $X_i\in S_i$ for every $i\in I$ and the set

$$\{i\in I : X_i\neq E_i\}$$

is finite.

The family of all rectangles X of E is denoted by the symbol P_0 and the family of all finite unions of rectangles - by the symbol P. Obviously, the family P is an algebra of subsets of E generated by P_0. We define a function

$$\mu : P_0 \longrightarrow R$$

by the formula

$$\mu(\prod_{i\in I}X_i)=\prod_{i\in I}\mu_i(X_i).$$

It is easy to verify that the function μ can be uniquely extended to a σ-additive function on the algebra P. We denote this extension by the same symbol μ. Hence, by Theorem 1.8, the measure μ can be extended to the uniquely termined measure on the σ-algebra $S=\sigma(P)$. The latter σ-algebra is denoted by $\prod\limits_{i\in I}S_i$. The extended measure defined on the σ-algebra

$\prod_{i\in I} S_i$ is denoted by $\prod_{i\in I}\mu_i$ and is called a product measure of the family of measures $(\mu_i)_{i\in I}$. The product of the family $(E_i,S_i,\mu_i)_{i\in I}$ is the measure space

$$(\prod_{i\in I}E_i,\prod_{i\in I}S_i,\prod_{i\in I}\mu_i).$$

Now, we recall another classical fact from measure theory, namely, the well-known Fubini theorem, which reduces the integration of real functions on the product measure space to the integration on the factors.

Theorem 1.15 (Fubini) *Let (E_1,S_1,μ_1) and (E_2,S_2,μ_2) be two measure spaces with σ-finite measures and let*

$$(E,S,\mu)=(E_1,S_1,\mu_1)\times(E_2,S_2,\mu_2).$$

Suppose that $f:E\longrightarrow\mathbf{R}$ is a μ-integrable function. Then:

1) for μ_1-almost every $x\in E_1$ the function

$$y\longrightarrow f(x,y)\ (y\in E_2)$$

is μ_2-integrable;

2) for μ_2-almost every $y\in E_2$ the function

$$x\longrightarrow f(x,y)\ (x\in E_1)$$

is μ_1-integrable;

3) the function

$$x\longrightarrow\int_{E_2}f(x,y)d\mu_2(y)$$

is μ_1-integrable and the function

$$y\longrightarrow\int_{E_1}f(x,y)d\mu_1(x)$$

is μ_2-integrable;

4) the equality

$$\int_{E_1}(\int_{E_2}f(x,y)d\mu_2(y))d\mu_1(x)=\int_{E_2}(\int_{E_1}f(x,y)d\mu_1(x))d\mu_2(y)=$$

$$=\int\int_{E_1\times E_2}f(x,y)d(\mu_1(x)\times\mu_2(y))$$

holds.

Analogously, one can formulate and prove the Fubini theorem for the product of finitely many measure spaces with σ-finite measures.

Let S be a given σ-algebra of subsets of a basic set E. A function

$$\nu : S \longrightarrow \overline{\mathbf{R}}$$

is called a signed measure on S if
a) $\nu(\emptyset) = 0$;
b) card(ran(ν) $\cap \{-\infty, +\infty\}) \leq 1$;
c) ν is σ-additive.

The next result actually reduces the notion of a signed measure to the usual notion of measure

Theorem 1.16 (Hahn) *Suppose that ν is a signed measure on a σ-algebra S of subsets of the basic set E. Then there exist two sets $A \subseteq E$ and $B \subseteq E$, such that:*
1) $A \cap B = \emptyset, A \cup B = E$;
2) $A \in S, B \in S$;
3) *for every $X \in S$ we have $\nu(A \cap X) \geq 0$ and $\nu(B \cap X) \leq 0$.*

The decomposition $\{A, B\}$ of the basic set E, corresponding to the given signed measure ν, is called a Hahn decomposition of E with respect to ν.

We define

$$\nu^+(X) = \nu(X \cap A) \ (X \in S),$$

$$\nu^-(X) = -\nu(X \cap B) \ (X \in S).$$

It is obvious that ν^+ and ν^- are ordinary measures on the σ-algebra S. Moreover, we have

$$\nu = \nu^+ - \nu^-.$$

Hence any signed measure ν can be represented as a difference of two ordinary measures. This representation is called a Jordan decomposition of the signed measure ν.

Note that the function $|\nu|$ defined by the formula

$$|\nu| = \nu^+ + \nu^-$$

is a measure on the σ-algebra S and is called a total variation of a given signed measure ν.

We say that a signed measure ν is σ-finite if there exists a family $(X_n)_{n \in N} \subseteq S$ such that $\cup_{n \in N} X_n = E$ and $|\nu|(X_n) < +\infty$ for every $n \in N$.

Suppose now that (E, S, μ) is a measure space and ν is a signed measure on the same σ-algebra S. We say that ν is absolutely continuous with respect to μ if

$$(\forall X \in S)(\mu(X) = 0 \longrightarrow \nu(X) = 0).$$

The next result plays an important role in modern analysis and the probability theory.

Theorem 1.17 (Radon-Nikodym) *Suppose that (E, S, μ) is a measure space with a σ-finite measure and ν is a σ-finite signed measure on S, absolutely continuous with respect to μ. Then there exists a μ- measurable function $f : E \rightarrow \mathbf{R}$ such that for every $X \in S$ we have*

$$\nu(X) = \int_X f d\mu.$$

The function f is called Radon-Nikodym derivative of the measure ν with respect to μ and is denoted by $\frac{d\nu}{d\mu}$.

Let E be an arbitrary topological space and let $B(E)$ be the Borel σ-algebra of this space. We say that a measure μ is a Borel measure (on E) if the equality

$$\mathrm{dom}(\mu) = B(E)$$

holds. It is obvious that the specific properties of the original topological space E often imply the corresponding specific properties of the Borel measures on E.

The following definition describes a very important class of Borel measures.

Let E be an arbitrary Hausdorff topological space and let μ be a Borel measure on E. We say that the measure μ is a Radon measure if for each set $X \in B(E)$ we have

$$\mu(X) = \sup\{\mu(K) : K \text{ is compact in } E \ \& \ K \subseteq X\}.$$

We say that a Hausdorff topological space E is a Radon space if every σ-finite Borel measure on E is a Radon measure. From the latter definition it immediately follows that any Borel subset of a Radon topological space is also a Radon space.

The following proposition is essentially due to Ulam (cf.[8],[86],[173]).

Theorem 1.18 *Any Polish topological space E is a Radon space.*

In context of Theorem 1.18 see Remark 12.8.

Let (E, S, μ) be a measure space.

A transformation $g : E \to E$ is called admissible (in the sense of invariance for measure μ)if

$$(\forall X)(X \in S \to g(X) \in S \ \& \ \mu(g(X)) = \mu(X)).$$

A transformation $h : E \to E$ is called admissible (in the sense of quasiinvariance for measure μ) if

$$(\forall X)(X \in S \to h(X) \in S \ \& \ (\mu(X) = 0 \iff \mu(h(X)) = 0)).$$

Note that an arbitrary admissible transformation in the sense of invariance is also an admissible transformation in the sense of quasiinvariance for an arbitrary nontrivial measure. In general, the converse proposition is not valid.

Another important mathematical structure closely connected with measures is a group structure.

Let E be a basic set and let G be some group of transformations of this set. Further, let D be some class of subsets of E. We say that the class D is G-invariant if

$$(\forall X)(\forall g)(X \in D \ \& \ g \in G \to g(X) \in D).$$

Let S be some σ-algebra of subsets of E and let μ be a measure defined on S. We say that the measure μ is G-quasiinvariant if:

1) S is a G-invariant class of subsets of E;

2) the class $L(\mu)$ of all μ-measure zero sets is also G-invariant.

If instead of the condition 2) the stronger condition

$$3)(\forall X)(\forall g)(X \in S \ \&\ g \in G \ \rightarrow \mu(g(X)) = \mu(X))$$

holds, then we say that the measure μ is G-invariant.

We say that a G invariant measure μ has the property of metrical transitivity (or is metrically transitive) if, for an arbitrary set $X \in \mathrm{dom}(\mu)$ with $\mu(X) > 0$, there exists a countable family $(g_k)_{k\in N}$ of transformations from G such that

$$\mu(E \setminus \cup_{k\in N} g_k(X)) = 0.$$

A subset $X \subseteq E$ is called an almost invariant set with respect to G-invariant (G-quasiinvariant) measure μ, if

$$(\forall g)(g \in G \rightarrow \mu(g(X)\Delta X) = 0).$$

A structure (E,S,G,μ) is called an invariant measure space if the following conditions hold:

1) E is a nonempty set;

2) G is some group of transformations of E;

3) S is some G-invariant σ-algebra of subsets of E;

4) μ is a G-invariant measure defined on S.

The following definition is important for the theory of invariant measures.

Let (E,S_1,G,μ_1) and (E,S_2,G,μ_2) be two invariant measure spaces such that $S_1 \subseteq S_2$. We say that μ_2 is a G-invariant extension of μ_1 if

$$(\forall X)(X \in S_1 \rightarrow \mu_2(X) = \mu_1(X)).$$

Remark 1.1 By using the well-known Carathéodory theorem, we can easily establish that usual extension of an arbitrary invariant σ-finite measure, defined on some invariant σ-ring of subsets of the basic space, is an invariant measure, too. Note that an analogous result, in general, is not valid for quasiinvariant measures. Indeed, if we consider the canonical Gaussian Borel measure λ in $\mathbf{R}^N$, then its restriction on the Baire algebra is $\mathbf{R}^N$-quasiinvariant whenever λ, following Kakutani's one result, is not $\mathbf{R}^N$-quasiinvariant.

Below, we will consider some properties of products of invariant (quasiinvariant) measures.

Let $(G_i)_{i\in I}$ be an arbitrary family of groups. Let e_i be a neutral element of the group G_i $(i \in I)$. The direct sum of the family of groups $(G_i)_{i\in I}$ is denoted by $\sum\limits_{i\in I} G_i$ and defined as

$$\sum_{i\in I} G_i = \{(g_i)_{i\in I} : (\forall i)(i \in I \rightarrow g_i \in G_i \ \&\ \mathrm{card}(\{i : g_i \neq e_i\}) < \omega)\}.$$

By using the Fubini theorem, one can obtain the validity of the following assertion.

Theorem 1.19 *Let $(E_i,S_i,G_i,\mu_i)_{i\in I}$ be a family of invariant (quasiinvariant) probability measure spaces. Then the product-measure $\prod_{i\in I}\mu_i$ is a $\sum_{i\in I} G_i$-invariant (quasiinvariant) probability measure.*

Observe that an analogue of Theorem 1.19 also holds for σ-finite invariant (quasiinvariant) measures when $\mathrm{card}(I) < \omega$.

Let $(G,.)$ be an arbitrary locally compact topological group. For such a group, we have the well-known result, belonging to Haar, which states the existence (and, in a certain sense, the uniqueness) of a nonzero invariant Borel measure μ on G (see e.g. [54]). Speaking more exactly, the measure μ satisfies the following relations:

1) μ is a locally finite measure, i.e., for each point $x \in G$ there exists an open neighborhood $V(x)$ of this point, such that $\mu(V(x)) < +\infty$;

2) μ is a Radon measure;

3) μ is a left invariant measure on $B(G)$, i.e.,

$$(\forall X)(\forall g)(X \in B(G) \ \&\ g \in G \ \rightarrow \mu(g \cdot X) = \mu(X)).$$

The measure μ is called a (left) Haar measure on the group G.

If the group G is σ-compact, then the Haar measure μ on G is a σ-finite measure. If the group G is compact, then the Haar measure μ is a finite measure, and we can obviously assume that, in this case, μ is a probability measure.

Chapter 2

Gaussian Measures

The theory of Brownian motion occupies an important place in modern science because it realizes a permanent and fruitful exchange of physical and mathematical problems (cf.[178]).

Brownian motion was discovered by botanist Brown in 1829 and was studied by many physicists in the 19-th century (see [119]), but the first satisfactory mathematical model was developed by Einstein in 1905. The basic hypothesis has been formulated for particles, moving along the real axis, as follows: if $x(t)$ is an abscissa of the particle at the instant t and $t_0 < t_1 < \cdots < t_n$, then the sequent differences $x(t_i) - x(t_{i-1})$ (for $1 \le i \le n$) are random Gaussian variables. We are not going to discuss in detail the important experimental works of G. Perrone which confirmed Einstein's theory. We only want to recall Perron's wonderful remark that the observation of the Brownian motion trajectory led him to an idea of such "mathematical functions"that have no derivatives. This remark served as the starting stimulus for Wiener.

The quite different ideas, originating in the kinetic theory of gas, has been developed by Boltzmann and Gyps between 1870 and 1900, as follows: let us consider some gas consisting of N molecules with mass m and with absolute temperature T. Let us denote by ν_i the velocity of the i-th molecule of the given gas $(1 \le i \le N)$. Note that the kinetic energy of this system is equal to

$$\frac{m}{2}(\nu_1^2 + \cdots + \nu_N^2) = 3NkT,$$

where k denotes the Boltzmann constant.

According to Gyp's ideas, the huge number of collisions between molecules does not allow us to find the molecule velocities and we ought to introduce the law of probability distribution on the sphere S with dimension $3N$ defined by the above equation. The "micro-canonical" condition is a statement that the probability measure p defined on the sphere S is invariant under the group of all rotations of the sphere S. On the other hand, the Maxwell law is a statement that the probability p is a Gaussian measure with variance $\frac{2kT}{m}$. We must say that Borel noted in 1914 that the Maxwell law is a direct consequence of Gyps' hypothesis and the property of a sphere when the number of molecules contained there is very large. He considered the sphere S as a subset of a multi-dimensional Euclidean space and probability p as an invariant measure under the group of all rotations of the sphere S. Using

one classical approach based on the Stirling formula, he proved that the projection of p on the real axis is an appropriate Gaussian measure. Later, these results were defined more precisely by Gato and Levy.

Let $m \geq 1$ be the integer number and r be the real positive number. Consider

$$S_{m,r} = \{(x_1, \cdots, x_m, 0, 0, \cdots) : x_1^2 + \cdots + x_m^2 = r^2\}.$$

Let us denote by $\sigma_{m,r}$ the probability measure defined on $S_{m,r}$ and being invariant under the group of all rotations of the given space. The main result of Gato and Levy can be formulated in modern terms as follows: the sequence of measures $(\sigma_{m,1})_{m\in N}$ is weakly convergent to the unit mass, concentrated at the initial point $(0, 0, \cdots)$; the sequence of measures $(\sigma_{m,\sqrt{m}})_{m\in N}$ is weakly convergent to the measure Γ defined by

$$d\Gamma(x_1, \cdots) = \prod_{i=1}^{\infty} d\nu(x_n),$$

where by ν is denoted the canonical Gaussian probability measure in $\mathbf{R}$.

The measure Γ plays the role of a Gaussian probability measure, when the dimension of the basic space is infinite. Levy hoped that it was possible to define the Gaussian probability measure in an arbitrary infinite-dimensional Hilbert space by using the inner form. Indeed, Levy and Wiener showed that the measure Γ is invariant (in some sense) under some group of automorphisms of the space ℓ_2, but unfortunately the space ℓ_2 of all square summable sequences $(x_1, x_2, \cdots, x_n, \cdots)$ has measure zero with respect to Γ. At present, it is known that we can confine ourselves to the Gaussian semi-measure defined in an infinite-dimensional Hilbert space.

The following essential advancement belongs to Wiener: if a reasonable Gaussian measure exists in an infinite-dimensional Hilbert space, then the required measure ω can be constructed on the space of all continuous functions by proceeding from a "weakly canonical distribution" (see [177]).

An initial construction of the Wiener measure based on the important formula of Gato and Levy is the following statement:

$$\Gamma = \lim_{m\to\infty} \sigma_{m\sqrt{m}}.$$

For arbitrary $m \geq 1$, denote by H_m the set of all functions which are defined on $T =]0;1]$ and have a constant value on each interval $]\frac{k-1}{m}; \frac{k}{m}]$ $(k = 1, 2, \cdots, m)$. Denote by π_m a probability Borel measure on the unit Euclidean sphere in R^m, which is invariant under the group of all rotations of this sphere.

Let f_m be an isomorphism between H_m and R^m, to which, for every function taking the value a_k on the interval $]\frac{k-1}{m}, \frac{k}{m}]$, there corresponds the vector $(a_1, a_2 - a_1, \cdots, a_m - a_{m-1})$. Denote by ω_m the measure defined on H_m and being the image of the measure π_m at the reflection f_m^{-1}. Wiener denoted by ω the limit of the family of measures $(\omega_m)_{m\in N}$.

More exactly, denote by H the space of all step-functions on T, equipped with the topology of uniform convergence. It is clear that

$$(\forall m)(m \in N \to H_m \subseteq H).$$

We may assert that, for an arbitrary uniformly continuous bounded function F on H there exists the limit

$$A(F) = \lim_{m\to\infty} \int\limits_{H_m} F(x)d\omega_m(x).$$

Using thin analysis of the oscillation phenomenon of the “heads or tails” game and the argument of compactness developed by Daniel, Wiener showed that Daniel’s theorem about the existence of a measure can be applied to this situation. He concluded that there exists a measure ω which is concentrated on the space $C(T)$ of all continuous functions on T and can be defined by

$$A(F) = \int\limits_{C(T)} F(x)d\omega(x).$$

Wiener proved that the measure ω satisfies Einstein’s hypothesis. In particular, the formula

$$\int\limits_{C(T)} f(x(t_1),\cdots,x(t_n))d\omega(x) = (2\pi)^{-\frac{n}{2}}\prod_{i=1}^{n}(t_i-t_{i-1})^{-\frac{1}{2}}\times$$

$$\times\int\limits_{\mathbf{R}^n} f(x_1,\cdots,x_n)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}\Big)dx_1\cdots dx_n$$

holds, where f is an arbitrary bounded continuous function on $\mathbf{R}^n$ and $0<t_0<t_1<\cdots<t_n$ (by the definition $x(0)=0$).

Various methods are presently available for the construction of Wiener measures. For example, Paley and Wiener use Fourier random series. Let, for an arbitrary sequence of real numbers $a=(a_n)_{n\geq 1}$ and for an arbitrary positive integer number $m\geq 0$, $f_{m,a}$ be the function defined on the interval $]0;1]$ by the formula

$$f_{m,a} = a_1 + 2\cdot\sum_{k=2}^{2^{m+1}}\frac{1}{\pi k}a_{k-1}\sin(\pi kt).$$

We can show that the sequence $(f_{m,a})_{m\in N}$ tends to some continuous function f_a for a Γ-almost sequence a and the Wiener measure is the usual image of the measure Γ under the mapping F defined almost everywhere by

$$F(a) = f_a.$$

Kac, Donsker and Erdös showed in 1950 why one can replace the spherical measures π_n defined in $\mathbf{R}^n$ by general measures in the initial construction of Wiener measures. Their results allow us to establish a close connection of the Wiener measure with the so-called limit theorems.

The Brownian motion is now regarded only as an important example of the Markov Process. One can also find Kac’s application of the Wiener measure to solve parabolic partial differential equations (investigations in this direction can be found in [31],[49]).

The problem connected with a finite sequence of random variables

$$X = (X_1,\cdots,X_n)$$

can, in fact, be easily solved by using the Charathéodory theorem when P_X, defined by

$$P_X = \{\omega : X(\omega) \in \prod_{i=1}^{n} [a_k; b_k]\},$$

is known for an arbitrary closed rectangle of R^n.

In various situations, the measure P_X has discrete support or density with respect to the Lebesgue measure. When we have an infinite sequence $(X_n)_{n\in N}$ of random variables, the law P_n of the partial sequence $(X_1, \cdots, X_n)$ is known for an arbitrary integer $n \geq 1$. These laws satisfy the consistency conditions under which the sequence $(P_n)_{n\in N}$ is a projective system of measures. The probability of events connected with infinite sequences of random variables was defined by using "usual" tending to limit in 1920. For example, the probability of an event that the "heads and tails" game will be finished, was assumed to be equal to the limit of probability of an event that this game will be over during no more n trials, as well $n \to \infty$. Usually, such a theory is not correct and we obtain a variety of "paradoxes" when a concrete random event may have different probabilities.

Steinhaus was probably the first of the mathematicians who felt the necessity of considering the projective system of measures $(P_n)_{n\in N}$ together with its limit. Daniel proved the existence of such projective limits in general cases in 1919. This result was probably not known in Europe. It was again obtained by Kolmogorov in [104], where the author formulated an axiomatic concept of probability theory. We must say that an important circumstance on which the proofs of Daniel and Kolmogorov are based is the compactness of the considered spaces.

The method of the construction of probability measures in general function spaces, elaborated by Kolmogorov, Feller and Dub in 1935, has difficulties of other kinds. Let us consider, for example, an interval T in $\mathbf{R}$ as the set of observation moments for some "stochastic process". A set of all possible trajectories is the product-space $\mathbf{R}^T$ which can be considered as a projective limit of the partial products $\mathbf{R}^H$, where H runs the class of all finite subsets in T. The projective system of measures is defined here in general cases. The Kolmogorov theorem gives the measure on $\mathbf{R}^T$, but this measure is defined on a σ-algebra smaller than the σ-algebra of all Borel subsets of $\mathbf{R}^T$. One version of the Kolmogorov construction, belonging to Kakutani [78], is devoted to measures on general topological spaces. In his method, μ_H is considered as the measure in $\overline{\mathbf{R}}^H$ concentrated on $\mathbf{R}^H$; the compact space $E = \overline{\mathbf{R}}^T$ is the projective limit of finite-dimensional product-spaces $(\overline{\mathbf{R}}^H)_{H\subset T}$, and we can define the measure μ in E as the projective limit of measures $(\mu_H)_{H\subset T}$. Unfortunately, his method has many inconveniences, because the elements in $\overline{\mathbf{R}}^T$ do not have the regularity properties. This phenomenon does not allow us to do steps to forward and to exclude parasitic values $\pm\infty$ appearing in the compactification process of the real axis $\mathbf{R}$. It may be correct if we induce the measure μ from $\overline{\mathbf{R}}^T$ on some subspace (for example, on $C(T)$ in the case of the Brownian motion). The main difficulty arises from the fact that, in general cases, functional spaces, even similar to usual spaces, are not μ-measurable in $\overline{\mathbf{R}}^T$, and the choice of such functional spaces may be very difficult.

Remark 2.1 By using the properties of thick nonmeasurable (in the sense of the Haar measure) subsets in $\overline{R}^T$, in Chapter 4 we will force above-mentioned inconveniences in order to construct the Borel product-measures in R^T when $\mathrm{card}(T) > \omega$).

A decisive step made in this direction by Prokhorov in 1956 in [153] influenced the theory of stochastic processes. Giving an axiomatic form to the methods applied by Wiener, he proved an important theorem about the existence of a projective limit for measures defined on a wide class of function spaces.

A more narrow class of projective systems of measures was considered by Bochner in 1947 (see [9]).

Remark 2.2 Many interesting and important methods of the existence of projective systems of measures are available at the present time. In Chapter 5 we will apply such technique to invariant Borel measure construction problem.

Remark 2.3 In Chapter 4 we will discuss some properties of so-called Gaussian processes which have been successfully used in various applied theoretical-probabilistic aspects(cf.[51]), in various topics of statistics, prognosis and filtration of random processes, in the theory of optimal management of solutions of differential equations indignated by random processes (cf.[111],[119],[178] and others). Here we focus on the property of equivalence (under translations) of Gaussian Baire measures in the functional space $\mathbf{R}^{\alpha}$ for an arbitrary nonempty parameter set α. We must say that a general problem of absolute continuity of Gaussian measures in various functional spaces itself is interesting and it is an important that has been studied for more than a half century ago by many people in various styles. For example, equivalence and orthogonality relations between two infinite direct product-measures has been investigated by S. Kakutani [79]. Similar dichotomies have revealed themselves in the study of Gaussian stochastic processes. C. Cameron and W.E. Martin proved in [21] that if one considers the measures induced on path space by a Wiener process on the unit interval, then if the variances of the processes are different the measures are orthogonal. This sort of result was generalized by U. Grenander in [52], starting from the viewpoint of statistical estimation, and utilizing a Karhunen representation for the processes involved. The conditions of absolute continuity and the formula of density of Gaussian measure under translation is obtained also in [52]. Wider sufficient conditions for orthogonality of the measures induced on path space by continuous Gaussian processes on the unite interval were obtained by G.Baxter[7]. R.H. Cameron and W.T. Martin [22] have recently investigated certain aspects of the Wiener integral and have obtained for instance a result which shows how the integral is transformed under the nonhomogeneous transformation-translation plus linear homogeneous transformation. They also examined the effect on the induced measures of taking certain types of affine transformations of a Wiener process. I.E. Segal extended their results in [158], made the situation more transparent by using his notion of “weak convegence”, and found the conditions for equivalence in more large class of function spaces. In [39] it is shown that the equivalence-or-orthogonality dichotomy holds in general for pairs of measures induced by Gaussian stochastic processes, and Segal’s necessary and sufficient conditions for equivalence are extended to cover the case of nonzero mean. A structure of admissible translations has been studied by T.S. Pitcher [150]. The measures with everywhere dense group of admissible translations has been considered by V.N. Sudakov [170]. The theory of Gaussian measures in linear spaces has been elaborated by A.M. Vershik [174]. Some results about absolute continuity of measures generated by Markov processes have been considered by I.I. Gihman and A.V. Skorohod [47]. Investigations of the absolute continuity of Gaussian measure under nonlinear transformations have been made by V. Baklan and A.D. Shatashvili

[4]. Various questions devoted to absolute continuity of measures in functional spaces are considered also in [48]. Finally, we want to indicate the well-known result of M. Talagrand [171] asserted that an arbitrary Baire Gaussian measure in $\mathbf{R}^T$ is τ-smooth for an arbitrary parameter set T.

We must say that above-mentioned notes consist of some comments which only touch upon the problem but is not based on a full bibliography or an elucidation of the history of all significant works in the theory of random processes which are elaborated to study main problems of Gaussian measures in various functional spaces.

Chapter 3

Dynamical Systems

Mathematical modelling is a science whose main goal is to construct such mathematical models that can fully represent quantitative behaviour of an observed process.

In a wide class of mathematical models we distinguish the so-called dynamical systems describing the behaviour of various physical, economic and social processes.

Let us recall the classical notion of a dynamical system due to Birkhoff (see [120]).

Let (X,ρ) be a metric space, and let $f(x,t)_{x\in X, t\in R}$ be a family of transformations of X, i.e., the condition $f(x,t)\in X$ holds for arbitrary $t\in \mathbf{R}$ and $x\in X$.

We say that the family $(f(x,t))_{x\in X, t\in \mathbf{R}}$ is a dynamical system if the family $(f(x,t))_{x\in X, t\in \mathbf{R}}$, considered as the mapping of two variables satisfies the following three conditions:

1) $f(x,0)=x$ for each element $x\in X$;

2) the mapping f is continuous with respect to the variables x and t;

3) if $x\in X, t_1\in \mathbf{R}$ and $t_2\in \mathbf{R}$, then $f(f(x,t_1),t_2)=f(x,t_1+t_2)$.

The parameter t is understood as time. The transformation

$$(f(\cdot,t))_{t\in\mathbf{R}} : X\to X$$

is called a motion of a dynamical system.

For concrete $x\in X$, the set

$$\{f(x,t) : t\in\mathbf{R}\}$$

is called a trajectory of the corresponding motion.

For concrete $x\in X$, the set

$$\{f(x,t) : T_1\le t_2\le T_2\},$$

where $-\infty\le T_1<T_2\le+\infty$, is called an arc of the trajectory denoted by

$$f(x;T_1;T_2).$$

If $-\infty<T_1<T_2<+\infty$, then the arc is called finite.

The positive number T_2-T_1 is called a time length of the corresponding arc $f(x;T_1;T_2)$.

Now let us assume that an element $x\in X$ is fixed.

If the condition

$$(\forall t)(t \in \mathbf{R} \to f(x,t+s) = f(x,t))$$

is fulfilled for some parameter $s \in \mathbf{R}$, then $(f(x,t))_{t\in\mathbf{R}}$ is called a periodical motion.

The smallest positive number T for which the condition

$$(\forall t)(t \in \mathbf{R} \to f(x,t+T) = f(x,t))$$

holds is called a period of the motion $(f(x,t))_{t\in\mathbf{R}}$.

Sometimes, we have situations where the condition

$$(\forall t)(t \in \mathbf{R} \to f(x,t) = x)$$

holds for a concrete point $x \in X$.

The point x, which corresponds to such a "motion", is called a rest point.

Clearly, periodical motion that does not have the smallest positive period coincides with a rest point.

Some motion $(f(\cdot,t))_{t\in\mathbf{R}}$ may also occur when

$$(\forall t_1)(\forall t_2)(-\infty < t_1 < t_2 < \infty \to f(x,t_1) \neq f(x,t_2))$$

for any $x \in X$.

Therefore, in the case of a dynamical system, we can distinguish three topological types of trajectories : 1) a point; 2) a simple closed arc; 3) a continuous and one-to-one image of the open interval $]0;1[$. The corresponding types of motions are: 1) rest; 2) periodical motion; 3) nonperiodical motion.

Remark 3.1 One can generalize the Birkhoff notion of a dynamical system for all topological vector spaces (X,τ) in terms of the convergence generated by the topology τ.

Example 3.1 The classical dynamical system theory considers motions defined by the system of ordinary differential equations

$$\frac{dx_i}{dt} = X_i(x_1,x_2,\cdots,x_n)\ (1 \le i \le n),$$

where the right-hand sides are the continuous functions of $x = (x_1,x_2,\cdots,x_n)$ in some closed domain D of an n-dimensional Euclidean "phase space."

A solution of the system is a set of functions of the form

$$x_1 = x_1(t),\ x_2 = x_2(t),\ \cdots,\ x_n = x_n(t)$$

which reduces the system to the identity.

A general solution of such a system contains n arbitrary constants.

To specify a unique solution, we can write the initial conditions as

$$x_1(0) = x_1^{(0)},\ x_2(0) = x_2^{(0)},\ \cdots,\ x_n(0) = x_n^{(0)}.$$

Cauchy proved that the above system has a solution

$$x_i = f_i(x_1^{(0)},\cdots,x_n^{(0)};t)\ (i = \overline{1,n}\ )$$

satisfying the initial conditions and this solution is unique if the right-hand sides of the system are continuous and possess first order finite partial derivatives with respect to the variables $x_1, x_2, \cdots, x_n$ for the values $t = t_0, x_1 = x_1^{(0)}, \cdots, x_n = x_n^{(0)}$.

Note that $x_i = f_i(x_1^{(0)}, \cdots, x_n^{(0)}; t)$ $(i = \overline{1,n}\,)$ can be considered as a motion of the classical dynamical system starting at the point

$$(x_1^{(0)}, \cdots, x_n^{(0)}).$$

As is well known, an arbitrary solution can be extended, when $t \to \pm\infty$, or it reaches, for the finite value $t = T$, the boundary of the domain D. An arbitrary solution

$$x_i = f_i(x_1^{(0)}, \cdots, x_n^{(0)}; t) \ (i = \overline{1,n}\,)$$

is a continuous function with respect to t and the coordinates of the initial point. Since the right-hand sides of this system do not depend on the parameter t, as the motion started at the point x at the instant t_1 reaches the point x_1, the motion started at x_1 at the instant t_2 reaches the point x_2. Hence the motion reaches the point x_2 at the instant $t_1 + t_2$. Now it is not difficult to verify that the function $f(x,t)$ defined by

$$f(x,t) = (f_i(x_1, x_2, \cdots, x_n; t))_{1 \le i \le n}$$

is a dynamical system in the sense of Birkhoff.

Example 3.2 Let (X, ρ) be an arbitrary metric space. Let us put

$$(\forall x)(\forall t)(x \in X \ \& \ t \in \mathbf{R} \to f(x,t) = x).$$

Note that every trajectory of the given dynamical system is a rest point.

Example 3.3 Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. Put

$$(\forall x)(\forall t)(x \in \mathbf{R}^n \setminus \{0\} \ \& \ t \in \mathbf{R} \to f(x,t) = e^t \cdot x).$$

It is obviously that $(f(x,t))_{t \in \mathbf{R}}$ is a nonperiodical motion (starting at the point x) for all $x \in \mathbf{R}^n \setminus \{0\}$.

Example 3.4 Now, let C be the two-dimensional space of complex numbers. Put

$$(\forall z)(\forall t)(z \in C \ \& \ t \in \mathbf{R} \to f(z,t) = z \cdot e^{it}).$$

Similar calculations show us that $(f(x,t))_{x \in X, t \in \mathbf{R}}$ is a dynamical system whose every motion is periodical.

Example 3.5 If $(X, ||.||)$ is an arbitrary infinite-dimensional Banach space, then the system $(f(x,t))_{x \in X, t \in \mathbf{R}}$ defined by

$$(\forall x)(\forall t)(x \in X \ \& \ t \in \mathbf{R} \to f(x,t) = x + a \cdot t),$$

where $a \in X$ and $||a|| \neq 0$, is an example of a dynamical system whose every motion is nonperiodical.

Let $(f(x,t))_{x\in X,t\in\mathbf{R}}$ be some dynamical system defined in a metric space (X,ρ). The Borel measure ν defined in X is called invariant under the group of all motions of the dynamical system $(f(x,t))_{x\in X,t\in\mathbf{R}}$ if for an arbitrary ν-measurable set $A\subseteq X$ the relation

$$(\forall t)(t\in\mathbf{R}\rightarrow\nu(f(A,t))=\nu(A))$$

holds.

N. Krylov and N. Bogolyubov constructed invariant probability Borel measures for dynamical systems defined in compact metrizable spaces (see [120]). Their construction is essentially based on some well-known facts of measure theory, which will be formulated below.

Recall that the family $(\nu_n)_{n\in N}$ of probability Borel measures defined in a compact metric space X is weakly convergent to the measure ν if the condition

$$\lim_{n\rightarrow\infty}\int\limits_X\varphi(x)d\nu_n=\int\limits_X\varphi(x)d\nu$$

holds for an arbitrary continuous function $\varphi:X\rightarrow\mathbf{R}$.

It is well known that the space $C(X)$ of all continuous functions defined on X and equipped with the metric ρ, where

$$(\forall\varphi_1)(\forall\varphi_2)(\varphi_1\in C(X)\ \&\ \varphi_2\in C(X)\rightarrow\rho(\varphi_1,\varphi_2)=\max_{x\in X}|\varphi_1(x)-\varphi_2(x)|\ ),$$

is a separable metric space.

Note that a functional

$$A:C(X)\rightarrow\mathbf{R}$$

defined by

$$(\forall\varphi)(\varphi\in C(X)\rightarrow A(\varphi)=\int\limits_X\varphi(x)d\nu(x)),$$

where ν is some fixed probability Borel measure in (X,ρ), is a normed linear positively definite functional.

Riesz and Radon proved that the converse proposition is also true. In particular, the following assertion is valid.

Theorem 3.1 *Let X be a compact metric space. Let $C(X)$ be the space of all continuous functions defined on X. Then for an arbitrary normed linear positively definite functional $A:C(X)\rightarrow\mathbf{R}$, there exists a probability Borel measure ν in X such that*

$$(\forall\varphi)(\varphi\in C(X)\rightarrow A(\varphi)=\int\limits_X\varphi(x)d\nu).$$

The proof of Theorem 3.1 can be found in [8].

We say that a family $(\mu_n)_{n\in N}$ of probability Borel measures is weakly compact if there exists a subsequence $(\mu_{n_k})_{k\in N}$ weakly converging to some probability measure μ in X.

The following proposition is due to Helly (see [8]).

Theorem 3.2 *An arbitrary countable family of Borel probability measures defined in an arbitrary compact metric space is weakly compact.*

Below, we will consider the proof of the famous result due to N. Krylov and N. Bogolyubov.

Theorem 3.3 *Let $(f(x,t))_{x\in X,t\in\mathbf{R}}$ be a dynamical system defined in a compact metric space X. Then there exists a probability Borel measure which is invariant under the group $(f(\cdot,t))_{t\in\mathbf{R}}$ $(x\in X)$.*

Proof. Let m be an arbitrary probability Borel measure defined in X. For concrete $\tau\in\mathbf{R}\setminus\{0\}$ and $\varphi\in C(X)$ we set

$$A_\tau(\varphi)=\tfrac{1}{\tau}\int_0^\tau dt\int_X\varphi(f(x,t))dm(x).$$

By Theorem 2.1, we easily obtain the existence of a probability Borel measure m_τ such that

$$\frac{1}{\tau}\int_0^\tau dt\int_X\varphi(f(x,t))dm(x)=\int_X\varphi(x)dm_\tau(x).$$

By Theorem 3.2 we can construct a sequence $(\tau_n)_{n\in N}$ with $\lim\limits_{n\to\infty}\tau_n=\infty$ and find a probability measure μ such that the family $(m_{\tau_n})_{n\in N}$ of probability Borel measures is weakly convergent to the measure μ. Therefore, the condition

$$\lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}dt\int_X\varphi(f(x,t))dm(x)=\lim_{n\to\infty}\int_X\varphi(x)dm_{\tau_n}(x)=\int_X\varphi(x)d\mu(x).$$

is fulfilled for an arbitrary continuous real function φ on X.

We have to show that the measure μ is invariant under the group $(f(\cdot,t))_{t\in\mathbf{R}}$.

Indeed, the measure μ is invariant under the group $(f(\cdot,t))_{t\in\mathbf{R}}$ if and only if the equality

$$\int_X\varphi(x)d\mu(x)=\int_X\varphi(f(x,t_0))d\mu(x)$$

holds for an arbitrary parameter $t_0\in\mathbf{R}$ and for an arbitrary continuous function $\varphi\in C(X)$.

The above relation remains true when $\varphi=\varphi_G$ is the indicator-function of an arbitrary open set $G\subseteq X$. Hence, having the validity of the relation

$$\int_X\varphi_G(x)d\mu(x)=\int_X\varphi_G(f(x,t_0))d\mu(x)$$

for an arbitrary open set G, we get

$$\int_X\varphi_G(x)d\mu(x)=\mu(G)=\int_X\varphi_G(f(x,t_0))d\mu(x)=\mu(f(G,-t_0)),$$

i.e.,

$$\mu(G)=\mu(f(G,-t_0)).$$

Note that this relation can be easily extended to the class of all measurable subsets of X.

Conversely, the validity of the condition

$$\mu(G) = \mu(f(G, -t_0))$$

for an arbitrary measurable subset $G \subseteq X$ and for an arbitrary parameter $t_0 \in \mathbf{R}$ implies the equality of the integrals $\int\limits_X \varphi(x) d\mu(x)$ and $\int\limits_X \varphi(f(x,t_0)) d\mu(x)$. This equality is equivalent to the condition

$$\lim_{n\to\infty} \frac{1}{\tau_n} \int\limits_0^{\tau_n} dt \int\limits_X \varphi(f(x,t)) dm(x) = \lim_{n\to\infty} \frac{1}{\tau_n} \int\limits_0^{\tau_n} dt \int\limits_X \varphi(f(x,t_0+t)) dm(x).$$

By the Fubini theorem, the functions under the limit operation can be rewritten as follows:

$$\int\limits_X dm(x) \frac{1}{\tau_n} \int\limits_0^{\tau_n} \varphi(f(x,t)) dt$$

and

$$\int\limits_X dm(x) \frac{1}{\tau_n} \int\limits_0^{\tau_n} \varphi(f(x,t_0+t)) dt.$$

Obviously, we get the following estimation:

$$|\frac{1}{\tau_n} \int\limits_0^{\tau_n} \varphi(f(x,t)) dt - \frac{1}{\tau_n} \int\limits_0^{\tau_n} \varphi(f(x,t+t_0)) dt| =$$

$$|\frac{1}{\tau_n} \int\limits_0^{\tau_n} \varphi(f(x,t)) dt - \frac{1}{\tau_n} \int\limits_{t_0}^{\tau_n+t_0} \varphi(f(x,t+t_0)) dt| =$$

$$|\frac{1}{\tau_n} \int\limits_{\tau_n-t_0}^{\tau_n+t_0} \varphi(f(x,t)) dt| \le \frac{2|t_0|M}{\tau_n},$$

where M is the maximum of $|\varphi|$ on X.

Note here that $M < \infty$ because φ is a continuous real function defined on the compact metric space X. Hence there exists a finite limit

$$\lim_{n\to\infty} \frac{1}{\tau_n} \int\limits_0^{\tau_n} dt \int\limits_X \varphi(f(x,t)) dm(x)$$

and for a sufficiently large natural number n, the module of the difference given above becomes less than an arbitrary small positive number ε, which completes the proof of the theorem. $\square$

In general cases, the behaviour of some dynamical systems can be characterized by the following important theorem.

Theorem 3.4 (Markov-Kakutani) *Let E be a Hausdorff topological vector space over* $\mathbf{R}$*, let T be a nonempty compact convex set in E, let U be a set of linear transformations of the space E into itself such that the conditions*

$$(\forall u)(\forall v)(u \in U \ \& \ v \in U \rightarrow u \circ v = v \circ u),$$

$$(\forall u)(u \in U \rightarrow u(T) \subseteq T),$$

$$(\forall u)(u \in U \rightarrow \textit{the restriction of } u \textit{ to } T \textit{ is continuous})$$

hold.

Then there exists a point $x_0 \in T$ such that

$$(\forall u)(u \in U \rightarrow u(x_0) = x_0).$$

The proof of Theorem 3.4 can be found e.g. in [85].

Corollary 3.1. *Krylov-Bogolyubov theorem is a direct consequence of the Markov-Kakutani theorem.*

Proof. Let us denote by $C(X)$ a vector space of all continuous functions

$$f : X \rightarrow \mathbf{R}$$

equipped with the usual norm

$$||f|| = \max_{x \in X} |f(x)|.$$

Obviously, $C(X)$ is a vector space over the field $\mathbf{R}$ of all real numbers. On the other hand, $C(X)$ is a Banach space under the norm defined above.

Taking into account the definition

$$f \leq f' - (\forall x)(x \in X \rightarrow f(x) \leq f'(x)),$$

we conclude that the space $C(X)$ is a partially ordered Banach space.

Let E be the space of all linear continuous functionals defined in $C(X)$. Note that E is a Hausdorff locally convex topological vector space over the field $\mathbf{R}$. Let us denote by T the subset of E consisting of linear functionals

$$\Phi : C(X) \rightarrow \mathbf{R}$$

which satisfy the two conditions:

$a)$ $(\forall \varphi)(\forall \varphi_1)(\varphi \in C(X) \ \& \ \varphi_1 \in C(X) \ \& \ \varphi \leq \varphi_1 \rightarrow \Phi(\varphi) \leq \Phi(\varphi_1))$,

$b)$ $\Phi(I_X) = 1$, where I_X is an indicator-function of the set X.

It is easy to see that T is a convex closed subset in E. Let us show that T is a compact subset in E.

If we set

$$\varphi^+(x) = \max_{x \in X}(\varphi(x), 0),$$

$$\varphi^-(x) = \min_{x \in X}(\varphi(x), 0)$$

for all functions $\varphi \in C(X)$, then

$$|\Phi(\varphi)| = |\Phi(\varphi^+) + \Phi(\varphi^-)| \leq \Phi(\varphi^+) + \Phi(-\varphi^-) \leq 2||\varphi||.$$

This means that

$$\Phi \in \prod_{\varphi \in C(X)} [-2||\varphi||, 2||\varphi||]_\varphi,$$

i.e.,

$$T \subseteq \prod_{\varphi \in C(X)} [-2||\varphi||, 2||\varphi||]_\varphi.$$

Following Theorem 2.4, the set

$$\prod_{\varphi \in C(X)} [-2||\varphi||, 2||\varphi||]_\varphi$$

is compact in $\mathbf{R}^{C(X)}$. Hence, taking into account the pointwise convergences in E and in $\mathbf{R}^{C(X)}$, respectively, we conclude that T is compact in E. Now let us verify that the set T is not an empty one. Indeed, if x_0 is an arbitrary element of X, then for all $\varphi \in C(X)$ we can put

$$\Phi_0(\varphi) = \varphi(x_0).$$

It is clear that Φ_0 is a linear positively definite continuous functional on $C(X)$ and $\Phi_0(I_X) = 1$. Since

$$\Phi_0 \in T,$$

we conclude that T is not empty.

Let us define $\varphi_t : X \to \mathbf{R}$ by the formula

$$\varphi_t(z) = \varphi(f(z,t)) \ (z \in X),$$

where $t \in \mathbf{R}, z \in X$ and $\varphi \in C(X)$.

Clearly, the relation

$$(\forall t)(t \in \mathbf{R} \to \varphi_t \in C(X))$$

is valid.

For arbitrary $\Phi \in E$ we set

$$(\forall t)(t \in \mathbf{R} \ \& \ \varphi \in C(X) \to \Phi_t(\varphi) = \Phi(\varphi_t)).$$

Define also the mapping

$$u_t : E \to E$$

for an arbitrary parameter $t \in \mathbf{R}$ by the formula

$$(\forall \Phi)(\Phi \in E \to u_t(\Phi) = \Phi_t).$$

Now it is easy to verify that the following conditions are satisfied:

$a)$ $(\forall t)(t \in \mathbf{R} \to u_t$ is a continuous linear mapping of E into itself$)$;

$b)$ $(\forall t)(t \in \mathbf{R} \to u_t(T) \subseteq T)$;

$c)$ $(\forall t_1)(\forall t_2)(t_1 \in \mathbf{R}\ \&\ t_2 \in \mathbf{R} \to u_{t_1} \circ u_{t_2} = u_{t_2} \circ u_{t_1})$;

d) The family $(u_t)_{t \in \mathbf{R}}$ of mappings is a group with respect to the usual composition operation, being isomorphic to the one-parameter group

$$(g_t)_{t \in R} = (f(\cdot, t))_{t \in \mathbf{R}}.$$

Indeed, we have:

Proof of $a)$**.**

$$u_t(\Phi_1 + \Phi_2)((\varphi(x))_{x \in X}) = (\Phi_1 + \Phi_2)_t((\varphi(x))_{x \in X}) = (\Phi_1 + \Phi_2)(\varphi((f(x,t))_{x \in X})) =$$

$$\Phi_1(\varphi((f(x,t))_{x \in X})) + \Phi_2(\varphi((f(x,t))_{x \in X})) = (\Phi_1)_t((\varphi(x))_{x \in X}) + (\Phi_2)_t((\varphi(x))_{x \in X}) =$$

$$u_t(\Phi_1)((\varphi(x))_{x \in X}) + u_t(\Phi_2)((\varphi(x))_{x \in X}).$$

Last relation means that $u_t : E \to E$ is linear. Let show that u_t is continuous. Indeed, let

$$\lim_{n \to \infty} ||\Phi_n((\varphi(x))_{x \in X}) - \Phi((\varphi(x))_{x \in X})|| = 0$$

for arbitrary $(\varphi(x))_{x \in X} \in C(X)$. Then we have

$$\lim_{n \to \infty} ||u_t(\Phi_n((\varphi(x))_{x \in X})) - u_t(\Phi((\varphi(x))_{x \in X}))|| =$$

$$\lim_{n \to \infty} ||\Phi_n(\varphi(f(x,t))_{x \in X}) - \Phi(\varphi(f(x,t))_{x \in X})|| = 0,$$

because $\varphi(f(x,t))_{x \in X} \in C(X)$. $\square$

Proof of $b)$. Let $\Phi \in T$. It means that

1) $(\forall \varphi_1)(\forall \varphi_2)(\varphi_1, \varphi_2 \in C(X)\ \&\ \varphi_1 \leq \varphi_2 \to \Phi(\varphi_1) \leq \Phi(\varphi_2))$;

2) $\Phi(I_X) = 1$.

We have

$$u_t(\Phi)((\varphi_1(x))_{x \in X}) = \Phi_t((\varphi_1(x))_{x \in X}) = \varphi_1((f(x,t))_{x \in X}) \leq \varphi_2((f(x,t))_{x \in X}) =$$

$$\Phi_t((\varphi_2(x))_{x \in X}) = u_t(\Phi)((\varphi_2(x))_{x \in X}).$$

$$u_t(\Phi)((I_X(x))_{x \in X}) = \Phi_t((I_X(x))_{x \in X}) = (I_X(f(x,t)))_{x \in X} = 1.$$

Last relation means that

$$u_t(\Phi) \in T.$$

Hence, we have proved that $u_t(T) \subset T$. $\square$

Proof of $c)$.

For $t_1, t_2 \in R$ we have

$$u_{t_1} \circ u_{t_2}(\Phi((\varphi(x))_{x \in X})) = u_{t_2}(u_{t_1}(\Phi((\varphi(x))_{x \in X}))) = u_{t_2}(\Phi_{t_1}((\varphi(x))_{x \in X})) =$$

$$u_{t_2}(\varphi((f(x,t_1))_{x \in X})) = \Phi_{t_2}(\varphi((f(x,t_1))_{x \in X})) = \Phi((\varphi(f(x,t_1),t_2))_{x \in X}) =$$

$$\Phi((\varphi(x,t_1+t_2))_{x\in X})=\Phi((\varphi(x,t_2+t_1))_{x\in X})=\Phi((\varphi(f(x,t_2),t_1))_{x\in X})=$$
$$\Phi_{t_1}((\varphi(f(x,t_2)))_{x\in X})=u_{t_1}(\Phi((f(x,t_2))_{x\in X}))=u_{t_1}(\Phi_{t_2}((\varphi(x))_{x\in X}))=$$
$$u_{t_1}(u_{t_2}(\Phi((\varphi(x))_{x\in X}))=u_{t_2}\circ u_{t_1}(\Phi((\varphi(x))_{x\in X}).$$

□

Proof of d)**.** We set
$$(\forall t)(t\in\mathbb{R}\to\Theta(u_t)=g_t).$$

We have
$$\Theta(u_{t_1}\circ u_{t_2})=\Theta(u_{t_1+t_2})=g_{t_1+t_2}=g_{t_2}\circ g_{t_1}.$$

□

Accordingly, we can imply the Markov-Kakutani theorem for T and $(u_t)_{t\in\mathbf{R}}$. Let $\overline{\Phi}$ be an element in T such that
$$(\forall t)(t\in\mathbf{R}\to u_t(\overline{\Phi})=\overline{\Phi}\ ).$$

The latter condition shows that
$$(\forall t)(t\in\mathbf{R}\to\overline{\Phi_t}=\overline{\Phi}\ ),$$

i.e.,
$$(\forall t)(\forall\varphi)(t\in\mathbf{R}\ \&\ \varphi\in C(X)\to\overline{\Phi}(\varphi_t)=\overline{\Phi}(\varphi)).$$

Since $\overline{\Phi}$ is a normed linear positively definite functional, the Riesz-Radon theorem implies that there exists a probability Borel measure μ such that
$$(\forall\varphi)(\varphi\in C(X)\ \to\overline{\Phi}(\varphi)=\int_X\varphi(x)d\mu).$$

On the other hand, the equality $\overline{\Phi}(g_t)=\overline{\Phi}(g)$ $(t\in\mathbf{R})$ means that the condition
$$(\forall\varphi)(\varphi\in C(X)\to\int_X\varphi(f(x,t))d\mu(x)=\int_X\varphi(x)d\mu(x)\ ).$$
holds.

Thus, to prove Corollary 3.1, it remains to repeat the final part of the proof of Theorem 3.3. □

If the group of all motions $(f(\cdot,t))_{t\in R}$ does not have rest points and trajectories are uncountable, then the measure μ constructed in Theorem 3.3, is diffused.

If we ignore the continuity of $(f(x,t))_{t\in R,x\in X}$ with respect to variable $x\in X$ and preserve only a measurability of $f:X\times R\to X$, then an analogue of Theorem 3.3 is not valid. In particular, the following result is valid.

Theorem 3.5. Let X be an uncountable compact metric space. Then there exists a family $(f(x,t))_{x\in X,t\in R}$ of two variables satisfies the following four conditions:

1) $f(x,0)=x$ for each element $x\in X$;
2) the mapping f is continuous with respect to the variable t;
3) the mapping f is measurable with respect to the variables x and t;
4) if $x\in X,t_1\in R$ and $t_2\in R$, then $f(f(x,t_1),t_2)=f(x,t_1+t_2)$

and there exists no a probability Borel measure μ which will be invariant under the group of all motions $(f(\cdot,t))_{t\in R}$.

Proof. Let Φ be a Borel isomorphism between X and R. We set:

$$(\forall x)(\forall t)(x\in X \ \&\ t\in R \rightarrow (f(x,t))_{t\in R} = \Phi^{-1}(\Phi(x)+t)).$$

Assume the contrary and let μ be a Borel probability measure on X which is invariant under the group of all motions $(f(\cdot,t))_{t\in R}$.

Let consider a set $Y=\Phi^{-1}([0,1[)$. Then, on the one hand, $(f(Y,k))_{k\in Z}$ is an infinite family of pairwise disjoint Borel subsets in X and

$$(\forall k)(k\in Z \rightarrow \mu(f(Y,k))=\mu(Y)).$$

The last relation contradicts the finiteness of μ and Theorem 3.5 is proved. $\square$

Remark 3.2 Note that for the dynamical systems constructed in examples 3.1, 3.2, 3.4, 3.5, we cannot apply the Krylov-Bogolyubov theorem (also the Markov-Kakutani theorem), but there exist non-trivial σ-finite Borel measures which are invariant under the group of all motions of the corresponding dynamical systems. It is clear that such a measure does not exist for the dynamical system constructed in Example 3.3. On the other hand, we can indicate a probability Borel measure (for example, a canonical Gaussian probability measure in $\mathbf{R}^2$) which satisfies the condition

$$(\forall A)(\forall t)(A\in B(X) \ \&\ t\in R \rightarrow (\mu(f(A,t))=0 \leftrightarrow \mu(A)=0)).$$

A measure satisfying the above condition is called quasiinvariant under the group of all motions generated by the dynamical system $f(x,t)_{x\in X,t\in\mathbf{R}}$.

Here we want to discuss some analogous questions related to the Krylov-Bogolyubov theorem on unit sphere in infinite-dimensional separable complex-valued Hilbert spaces. Briefly speaking, we consider some statements of functional analysis and measure theory, which are frequently applied in constructions of invariant (under one-parameter group of unitary operators) measures in such spaces.

Here we introduce some notations:

$\mathbf{C}$ - complex numbers space;

$\mathbf{R}^N$- a topological vector space of all real-valued sequences, equipped with the Tykhonoff topology;

$\mathbf{C}^N=\prod_{k\in N}\mathbf{C}_k$, where $\mathbf{C}_k=\mathbf{C}$ for $k\in N$;

$\mathbf{C}^{(N)}=\{(z_k)_{k\in N}: z_k\in\mathbf{C} \ \&\ \mathrm{card}(\{k: z_k\neq 0\})\leq\aleph_0\}$;

$W^2=\{(z_k)_{k\in N}: z_k\in\mathbf{C}, k\in N, \sum_{k\in N}|z_k|^2<+\infty\}$ - an infinite-dimensional separable complex-valued Hilbert space equipped with the usual inner scalar product $<\cdot,\cdot>_1$ defined by

$$(\forall (z_k)_{k\in N}\ ,\ (w_k)_{k\in N}\in W^2 \rightarrow <(z_k)_{k\in N},(w_k)_{k\in N}>_1=\sum_{k\in N} z_k\times\overline{w_k}),$$

where $\overline{w_k}$ denotes the conjugate of the complex number w_k $(k\in N)$;

$S^2=\{(z_k)_{k\in N}:(z_k)_{k\in N}\in W_2 \ \&\ \sum_{k\in N}|z_k|^2=1\}$ - the unit sphere in W^2;

$B(W^2)$ - the σ-algebra of all Borel subsets of W^2;

$B(S^2)$ - the σ-algebra of all Borel subsets of S^2;
$\mathbf{R}^n$ - the n-dimensional Euclidean vector space;
$B(\mathbf{R}^n)$ - the σ-algebra of all Borel subsets of $\mathbf{R}^n$;
$L^2(\mathbf{R}^3,\mathbf{C})$ - the infinite-dimensional separable complex-valued Hilbert space defined by

$$L^2(\mathbf{R}^3,\mathbf{C}) = \{ \Psi | \Psi : \mathbf{R}^3 \to \mathbf{C} \ \& \int_{\mathbf{R}^3} |\Psi(x)|^2 dx < +\infty \} ,$$

equipped with the usual inner scalar product $< \cdot,\cdot >_2$ defined by

$$(\forall \Psi_1,\Psi_2 \in L^2(\mathbf{R}^3,\mathbf{C}) \to < \Psi_1,\Psi_2 >_2 = \int_{R^3} \Psi_1(x)\overline{\Psi_2}(x)dx),$$

where $\overline{\Psi_2}$ denotes the conjugate of the complex-valued function Ψ_2.
S^{*2} - the unit sphere in $L^2(\mathbf{R}^3,\mathbf{C})$.
$||\cdot||_1$ and $||\cdot||_2$ denote usual norms in W^2 and in $L^2(\mathbf{R}^3,\mathbf{C})$, respectively.

Let $(f(\cdot,t))_{t\in\mathbf{R}}$ be one-parameter group of unitary operators on W^2. Here we recall the well-known result of Stone[1] (cf.[94],[154]) giving information about the structure of such groups.

Theorem 3.6 (Stone) *For an arbitrary one-parameter group $(f(\cdot,t))_{t\in\mathbf{R}}$ of unitary operators in W^2 there exists a Hermitian[2] operator A in W^2 such that*

$$f(\cdot,t) = e^{itA}(\cdot) \ (t \in \mathbf{R}).$$

As the unite sphere S^2 does not satisfy conditions in theorems 3.3 and 3.4, here naturally the following arises

Problem 3.1 *Let A be an arbitrary Hermitian operator on W^2. Does there exist a Borel probability measure on S^2 which is invariant under the one-parametric group of unitary operators $(e^{itA})_{t\in\mathbf{R}}$ on W^2 ?*

Remark 3.3 We can easily demonstrate that on the unit sphere S^2, there exists no probability Borel measure invariant under the group of all unitaryy operators in W^2. Indeed, assume the contrary and let p be a such measure. Denote by $(\Psi_k)_{k\in N}$ any orthonormal basis in W^2. It is clear that $||\Psi_k-\Psi_n|| = \sqrt{2}$ for $k,n\in N$ and $k\neq n$. Let $U(\Psi,r)$ be a spherical neighborhood of a point $\Psi\in S^2$ with $r<\frac{\sqrt{2}}{2}$. On the one hand, we must have that $p(U(\Psi,r)) = 0$ for an arbitrary $\Psi\in S^2$. Indeed, if we assume the contrary and $p(U(\Psi,r)>0$, then from the invariance of the measure p under the group of all unitaryy operators , we conclude that

$$p(\cup_{k\in N}U(\Psi_k,r)) = \sum_{k\in N} p(U(\Psi_k,r)) = +\infty,$$

since $(U(\Psi_k,r))_{k\in N}$ is a family of disjoint Borel sets in S^2 every element of which is an image of the set $U(\Psi,r)$ under action of some unitary operator. Note that last relation contradicts the condition $p(S^2) = 1$.

[1] A linear operator $U : W^2 \to W^2$ is called unitary if $< U(z_1),U(z_2) >_1 = < z_1,z_2 >_1$ for every $z_1,z_2 \in W^2$.
[2] A linear operator $A : W^2 \to W^2$ is called Hermitian if $< U(x),y >_1 = < x,U(y) >_1$ for every $x,y \in W^2$.

Now, let M be a everywhere countable dense subset in S^2. On the one hand, we have

$$S^2 = \cup_{\Psi \in M} U(\Psi, r).$$

On the other hand, we must have

$$p(S^2) = p(\cup_{\Psi \in M} U(\Psi, r)) \leq \sum_{\Psi \in M} p(\Psi, r) = 0,$$

which contradicts the condition $p(S^2) = 1$ and the impossibility of the definition of the Borel measure with the above-mentioned properties is showed. □

One can extend above-mentioned fact as follows:

There exists no probability Borel measure on S^2 which is quasiinvariant under the group of all unitaryy operators in W^2.

Now let, for $k \in N$, r_k denotes a such positive number that

$$\int\int_{x_1^2+x_2^2 \leq r_k^2} \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 = e^{-\frac{1}{2^k}}.$$

Let, for $k \in N$, μ_k denotes a such two-dimensional Gaussian Borel measure in $\mathbf{R}_k^2$ whose density has the form

$$\frac{1}{2\pi\delta_k^2} e^{-\frac{x_1^2+x_2^2}{2\delta_k^2}},$$

where $\mathbf{R}_k = \mathbf{R}$, $\delta_k = \frac{1}{2^k r_k}$.

In the sequel we need some auxiliary results.

Lemma 3.1 *We have*

$$\sigma(Rec(\mathbf{C}^N) \cap (\cup_{k \in N} B_n)) = \sigma(Rec(W^2) \cap (\cup_{n \in N} B_n))$$

$$= B(\mathbf{C}^N) \cap (\cup_{n \in N} B_n) = B(W^2 \cap (\cup_{n \in N} B_n)),$$

where $Rec(\cdot)$ and $B(\cdot)$ denote the algebra of all cylindrical sets and the Borel σ-algebra of the subsets of corresponding spaces, respectively .

The proof of Lemma 3.1 relies on direct checking of the equality above, and therefore is left to the reader.

Lemma 3.2 *Let E be a basic space, S be a G-invariant algebra of subsets of E, and μ be a G-invariant probability measure defined on S. Then the extension $\overline{\mu}$ of the measure μ to the minimal σ-algebra $\sigma(S)$ (generated by S) is a G-invariant probability measure.*

Remark 3.4 The Lemma 3.2 is a simple consequence of the well-known Carathéodory theorem. Note that analogous result, in general, is not valid for quasiinvariant σ-finite measures. Indeed, let μ_1 be the restriction of the canonical Gaussian measure μ in $\mathbf{R}^N$ to the algebra of all rectangular sets. One can easily demonstrate that μ_1 is quasiinvariant

under the group of all translations of $\mathbf{R}^N$, whenever, according to Kakutani's well-known result(cf.[31]), the group of all admissible (in the sense of quasiinvariance) translations for the measure μ coincides with ℓ_2.

Lemma 3.3 *Let $(E_i, S_i, G_i, \mu_i)_{1 \le i \le n}$ be a family of invariant measure spaces with invariant σ-finite measures. Then $\prod_{1 \le i \le n} \mu_i$ is a $\prod_{1 \le i \le n} G_i$-invariant measure.*

The proof of Lemma 3.3 can be obtained by using Fubini theorem.

Theorem 3.7 *Let*

$$(\forall X)(X \in B(R_k^2) \rightarrow \mu_k^c(X^c) = \mu_k(X)),$$

where $X^c = \{ x + iy : (x,y) \in X \}$.

Then, for the measure $\prod_{k \in N} \mu_k^c$, the following conditions are fulfilled:

1) $\prod_{k \in N} \mu_k^c(W^2) = 1$;

2) The measure $\prod_{k \in N} \mu_k^c$ is invariant under action of an arbitrary linear transformation $A : W^2 \rightarrow W^2$ whose corresponding matrix in the standard basis is a diagonal matrix $(a_{km})_{k,m \in N}$ with $a_{kk} = e^{id_k} (k \in N, d_k \in \mathbf{R})$.

Proof. Let

$$B_{0,2^{-k}} = \{ (x,y) : (x,y) \in \mathbf{R}_k^2 \ \& \ x^2 + y^2 \le 2^{-2k} \},$$

$$(\forall n)(n \in N \rightarrow B_n = \prod_{k=1}^{n} \mathbf{C}_k \times \prod_{k > n} B_{0,2^{-k}}).$$

It is clear that

$$(\forall n)(n \in N \rightarrow B_n \in B(W^2)).$$

Hence, $\cup_{n \in N} B_n \in B(W^2)$.

Let us show that $\prod_{n \in N} \mu_n^c(\cup_{n \in N} B_n) = 1$.

It is clear that $(\forall n)(n \in N \rightarrow B_n \subset B_{n+1})$.

By the property of lower semicontinuity of probability measures, it is sufficient to show that $\lim_{n \rightarrow \infty} \mu(B_n) = 1$.

Indeed, using the property of continuity from the left of the function e^x at the point $x = 0$, for an arbitrary positive number ε we can choice a positive number δ such that

$$(\forall x)(-\delta \le x \le 0 \rightarrow 1 - \varepsilon \le e^x \le 1).$$

Let n_ε be a natural number such that

$$-\delta \le -\sum_{k \ge n_\varepsilon} \frac{1}{2^k} \le 0.$$

Then, for $n \ge n_\varepsilon$, we obtain

$$\prod_{k\in N}\mu_k^c\ (B_n)=\prod_{k=1}^{n}\mu_k^c\ (\prod_{k=1}^{n}\mathbf{C}_k)\times\prod_{k>n}\mu_k^c\ (\prod_{k>n}B_{0,2^{-k}})$$

$$=\prod_{k>n}e^{-\frac{1}{2^k}}=e^{-\sum\limits_{k>n}\frac{1}{2^k}}.$$

Hence

$$1-\varepsilon<\prod_{k\in N}\mu_k^c\ (B_n)<1,$$

and the validity of the equality

$$\lim_{n\to\infty}\prod_{k\in N}\mu_k^c\ (B_n)=1$$

is proved.

Note that, on the one hand, we have

$$\prod_{k\in N}\mu_k^c\ (W^2)\leq 1$$

and, on the other hand, the relation

$$\cup_{k\in N}B_k\subseteq W^2$$

implies

$$\prod_{k\in N}\mu_k^c\ (W^2)\geq\prod_{k\in N}\mu_k^c\ (\cup_{k\in N}B_k)=1.$$

Finally, we get $\prod\limits_{k\in N}\mu_k^c\ (W^2)=1$ and the validity of the condition 1) in Theorem 3.7 is proved.

2) Let us show that the measure $\prod\limits_{k\in N}\mu_k^c$ is invariant under the transformation **A**.

According to lemmas 3.1 and 3.2 , the condition 2) in Theorem 3.7 will be proved if we show that the $\prod\limits_{k\in N}\mu_k^c$-measure of the set $X\times\prod\limits_{k>n}B_{0,2^{-k}}$ is invariant under the action of the transformation A for all $X\in B(\mathbf{C}^n)$.

Indeed,

$$\prod_{k\in N}\mu_k^c\ (A(X\times\prod_{k>n}B_{0,2^{-k}}))=\prod_{k\in N}\mu_k^c\ (A_n(X)\times\prod_{k>n}e^{id_k}(B_{0,2^{-k}}))$$

$$=\prod_{k\in N}\mu_k^c\ (A_n(X)\times\prod_{k>n}B_{0,2^{-k}})=\prod_{k=1}^{n}\mu_k^c\ (A_n(X))\times\prod_{k>n}\mu_k^c\ (\prod_{k>n}B_{0,2^{-k}})$$

$$=\prod_{k=1}^{n}\mu_k^c\ (X)\times\prod_{k>n}\mu_k^c\ (\prod_{k>n}B_{0,2^{-k}})=\prod_{k\in N}\mu_k^c\ (X\times\prod_{k\in N}B_{0,2^{-k}}),$$

where $A_n=(a_{km})_{1\leq k,m\leq n}$ is an n-dimensional diagonal matrix with $a_{kk}=e^{id_k},\ 1\leq k\leq n$.

This ends the proof of Theorem 3.7. $\square$

Let $X \in B(S^2)$. We say that a set $I_X \subseteq W^2$ is the X-cone if

$$I_X = \{ \Psi : \Psi \in W^2 \ \& \ (\exists \alpha)(\alpha \in \mathbf{R}^+ \to \alpha\Psi \in X) \} .$$

Lemma 3.4 $(\forall X)(X \in B(S^2) \to I_X \in B(W^2))$ *and the class* $K(W^2)$ *of all conical subsets in* W^2 *is an* A*-invariant* σ*-algebra of subsets of* $W^2 \setminus \{ \mathbf{0} \}$ *, where* A *is the same transformation defined by the condition* $2)$ *in Theorem* 3.7 *and* $\mathbf{0}$ *is a zero of* W^2.

Proof. Let us denote by G the class of all open sets in S^2. It is clear that $\sigma(G) = B(S^2)$. Let us show that, for arbitrary $Y \in G$, the Y-cone $I_Y \in B(W^2)$.
Indeed, let $z_0 \in I_Y$. This means that there exists $\alpha \in \mathbf{R}^+$ such that $\alpha z_0 \in Y$.
There are possible two cases: $0 < \alpha \leq 1$ or $\alpha > 1$.
Case 1. If $0 < \alpha \leq 1$, then $|z_0| \geq 1$ and there exists r $(0 < r < \frac{1}{2})$ such that

$$\{ z : z \in S^2 \ \& \ ||z - z_0||_1 \leq r \} \subset Y.$$

It follows that

$$\{ z : z \in W^2 \ \& \ ||z - z_0||_1 < r \} \subset I_Y.$$

Case 2. If $\alpha > 1$, then $|z_0| < 1$ and

$$\{ z : z \in W^2 \ \& \ ||z - z_0||_1 < \frac{|z_0| r}{2}\sqrt{4 - r^2} \} \subset I_Y.$$

Lemma 3.4 is proved. $\square$

Theorem 3.8 *The functional* μ *defined by*

$$(\forall X)(X \in B(S^2) \to \mu(X) = \prod_{k \in N} \mu_k^c(I_X)),$$

is an A*-invariant Borel probability measure defined on* S^2*, where* A *is the same transformation mentioned in Theorem* 3.7.

Proof. The functional μ is a probability Borel measure. Indeed, if $(X_k)_{k \in N}$ is the family of disjoint Borel subsets in S^2, then the family $(I_{X_k})_{k \in N}$ is also a disjoint family of Borel subsets in W^2 and

$$\mu(S^2) = \prod_{k \in N} \mu_k^c \ (I_{S^2}) = \prod_{k \in N} \mu_k^c \ (W^2 \setminus \{ \mathbf{0} \}) = \mathbf{1},$$

$$(\forall X)(X \in B(S^{*2}) \to \mu(X) = \prod_{k \in N} \mu_k^c \ (I_X) \geq 0),$$

$$\mu(\cup_{m \in N} X_m) = \prod_{k \in N} \mu_k^c \ (I_{\cup_{m \in N} X_m}) = \sum_{m=1}^{\infty} \prod_{k \in N} \mu_k^c \ (I_{X_m}) = \sum_{m=1}^{\infty} \mu(X_m).$$

Let us show that the measure μ is A-invariant. Indeed,

$$(\forall X)(X \in B(S^2) \to \mu(A(X)) = \prod_{k \in N} \mu_k^c \ (I_{A(X)}) =$$

$$\prod_{k\in N}\mu_k^c\ \ (A(I_X)) = \prod_{k\in N}\mu_k^c\ \ (I_X) = \mu(X)).$$

This ends the proof of Theorem 3.8. □

Let recall the following well-known result from functional analysis.

Theorem 3.9 (Hilbert-Schmidt) *If A is a bounded by the norm Hermitian operator defined in the infinite-dimensional complex-valued separable Hilbert space H, then for an arbitrary element $x \in H$ the element Ax may be represented as the Fourier series of its proper vectors.*

Corollary 3.2 If A is a bounded by the norm Hermitian operator defined in the complex-valued separable Hilbert space H, then there exists an orthonormal basis in H whose every element is a proper vector of the operator A (cf.[94]).

In the sequel we consider so-called normal Hermitian operators which have the property that there exists an orthonormal basis in H consisting only of its proper vectors. Note here that no every normal Hermitian operator is bounded by the norm in H.

The main result of the present chapter is contained in the following proposition.

Theorem 3.10 *Let A be a normal Hermitian operator in W^2. Then there exists such a Borel probability measure ν on the unite sphere S^2 which is invariant under the one-parametric group of unitaryy operators $(e^{itA})_{t\in\mathbf{R}}$.*

Proof. Let $(\lambda_k)_{k\in N}$ be a family of all proper numbers of the operator $\mathbf{A}$. According to Corollary 3.2, we conclude that there exists an orthonormal basis $(\Psi_k)_{k\in N}$ in W^2 consisting of the corresponding proper vectors of the operator A.

We define the functional ν by

$$(\forall X)(X \in B(S^2) \rightarrow \nu(\ \{\ \Psi : \Psi \in W^2\ \&\ (<\Psi, \Psi_k>_1)_{k\in N} \in X\ \}\ ) = \mu(X)).$$

It is clear that the probability measure ν is defined on the Borel σ-algebra of subsets of the unit sphere S^2 in W^2. A matrix defined by the operator A coincides with the infinite diagonal matrix $(a_{ij})_{i,j\in N}$ in the basis $(\Psi_k)_{k\in N}$ such that $a_{ii} = \lambda_i$. Hence $e^{itA} = (b_{mn})_{m,n\in N}$ is also a diagonal matrix with $b_{mm} = e^{it\lambda_m}$ $(m \in N)$.

Let us show that the measure ν is invariant under $(e^{itA})_{t\in\mathbf{R}}$.

Indeed, for arbitrary $X \in B(S^2)$, we have

$$\nu(e^{itA}\ \{\ \Psi : (<\Psi, \Psi_k>_1)_{k\in N} \in X\ \}\ ) = \nu(\ \{\ e^{itA}\Psi : (<\Psi, \Psi_k>_1)_{k\in N} \in X\ \}\ )$$

$$= \nu(\ \{\ \Psi^* : (<e^{itA}\Psi^*, \Psi_k>_1)_{k\in N} \in X\ \}\ )$$

$$= \nu(\ \{\ \Psi^* : (<\Psi^*, \Psi_k>_1)_{k\in N} \in e^{itA}(X)\ \}\ ) = \mu(e^{itA}(X)) = \mu(X)$$

$$= \nu(\ \{\ \Psi : (<\Psi, \Psi_k>_1)_{k\in N} \in X\ \}\ ).$$

Theorem 3.10 is proved. □

Corollary 3.3 *Let us consider the Schrödinger equation*

$$i\hbar\frac{d\Psi}{dt} = A\Psi,$$

where A is a normal Hermitian operator in $L^2(\mathbf{R}^3,\mathbf{C})$, $\Psi \in S^{*2}$, $\hbar$ is Plank's reduced constant.

Then there exists a Borel probability measure in S^{*2} which is invariant under the one-parametric group of unitary operators $(\ e^{-\frac{i}{\hbar}tA}\ )_{t\in\mathbf{R}}$.

Remark 3.5 As we know, Problem 3.1 remains open for the one-parametric group of unitary operators generated by the non-normal Hermitian operators.

Now, we are going to discuss the following important notion.

The dynamical system $(f(x,t))_{x\in X, t\in R}$ with an invariant (or quasiinvariant) measure μ defined on some invariant (under the group of motions) σ-algebra of subsets of X, is called a dynamical (quasi-dynamical) system in the wide sense.

Remark 3.6 Various equivalent definitions of the ergodicity of dynamical systems are presently available (for example, see [46], [49]).

By using similar methods of the theory of invariant and quasiinvariant measures, we can obtain the following classical results due to Poincaré and Carathéodory.

Theorem 3.11 (Recurrence of a Subset) *Let μ be a probability measure defined in a metric compact space X and being invariant under the group of all motions of the dynamical system $(f(x,t))_{x\in X,t\in R}$. If $A \in dom(\mu)$ and $\mu(A) > 0$, then there exists an instant t such that*

1) $|t| > 0$.

2) $\mu(A\cap f(A,t)) > 0$.

Proof. We set

$$B = \{x : x \in A \ \&\ (\forall t)(|t| > 0 \to f(x,t) \notin A)\}.$$

One can easily verify that $B \in B(X)$.

Let us show that $\mu(B) = 0$.

We begin by noting that

$$f(B,t_1)\cap f(B,t_2) = \emptyset$$

for all distinct $t_1,\ t_2 \in \mathbf{R}$.

Indeed, if we assume the contrary, then

$$f(B,t_1)\cap f(B,t_2) \neq \emptyset$$

for some distinct nonzero $t_1,\ t_2 \in \mathbf{R}$.

Let $y \in f(B,t_1)\cap f(B,t_2)$. Then

$$y \in f(f(B,t_1),-t_1)\cap f(f(B,t_2),-t_1) = f(B,0)\cap f(B,t_2-t_1) = B\cap f(B,t_2-t_1).$$

But the last relation means that for $y \in B$ there exists an element $x \in B$ such that

$$y = f(x,t_2-t_1),$$

i.e.,

$$f(y,t_1-t_2) = f(x,0) = x.$$

Thus, we obtain a contradiction with the definition of the set B.

Now, if we consider an infinite sequence of distinct instants $(t_k)_{k\in N}$, then

$$(f(B,t_k))_{k\in N}$$

will be an infinite family of disjoint images of the set B. If we assume that $\mu(B) > 0$, then we obtain a contradiction to the finiteness of the measure μ. Indeed, on the one hand we have,

$$\mu(X) = 1 < +\infty.$$

On the other hand,

$$\mu(X) \geq \mu(\cup_{k\in N} f(B,t_k)) = \sum_{k\in N} \mu(f(B,t_k)) = +\infty.$$

We obtain a contradiction, and the validity of the relation $\mu(B) = 0$ is proved. Consequently, $\mu(X \setminus B) = 1$, i.e.,

$$\mu(\{x : x \in A \ \& \ (\exists t)(|t| > 0 \to f(x,t) \in A)\}) = \mu(X) = 1.$$

The proof of Theorem 3.11 is completed. $\square$

We say that a point x is stable in the sense of Poisson if for an arbitrary neighbourhood U of x and for arbitrary $T > 0$ there exist $t_1 > T$ and $t_2 < -T$ such that $f(x,t_1) \in U$ and $f(x,t_2) \in U$, respectively.

Theorem 3.12 (Recurrence of Points) *Let X be a metric compact space, and let μ be a Borel probability measure being invariant under the group of all motions of the dynamical system $(f(x,t))_{x\in X, t\in R}$. Then μ-almost every point of X is stable in the sense of Poisson; moreover,*

$$(\forall T)(\forall A)(T > 0 \ \& \ A \in B(X) \to \mu(\{x : x \in A \ \& \ (\forall t)(|t| > T \ \& \ f(x,t) \notin A)\}) = 0),$$

where $B(X)$ denotes the σ-algebra of all Borel subsets of X.

Proof. Let T be an arbitrary positive number. Let U be an arbitrary open set in X. Note that U can be considered as a neighbourhood of its points. Define

$$B = \{x : x \in U \ \& \ (\forall t)(|t| > T \to f(x,t) \notin U)\}.$$

First, let us show that

$$(\forall t_1)(\forall t_2)(|t_1 - t_2| > T \to f(B,t_1) \cap f(B,t_2) = \emptyset).$$

Indeed, if we assume the contrary, then there exist $t_1 \in \mathbf{R}, t_2 \in \mathbf{R} \ \& \ t_1 - t_2 > T$, such that

$$f(B,t_1) \cap f(B,t_2) \neq \emptyset.$$

Hence

$$f(f(B,t_1),-t_1) \cap f((B,t_2),-t_1) \neq \emptyset \to B \cap f(B,t_2 - t_1) \neq \emptyset.$$

If $y \in B \cap f(B, t_2 - t_1)$, then there exists $x \in B$ such that $y = f(x, t_2 - t_1)$. Consequently,

$$x = f(y, t_1 - t_2)$$

and we obtain a contradiction with the condition

$$y \in B \,\&\, (\forall t)(|t| > T \to f(x,t) \notin A),$$

so that $f(y, t_1 - t_2) \in U$ and $|t_1 - t_2| > T$.

Now, we are going to show that $\mu(B) = 0$.

Indeed, if we assume the contrary, then $(f(B, 2kT))_{k \in Z}$ must be an infinite countable disjoint family of images of the set B, and from the invariance of μ we obtain

$$1 = \mu(X) \geq \mu(\cup_{k \in Z} f(B, 2kT)) = \sum_{k \in Z} \mu(f(B, 2kT)) = +\infty.$$

This is a contradiction and Theorem 3.12 is proved. $\square$

The following proposition is of some interest.

Theorem 3.13 *Let* (X, ρ) *be a metric space equipped with the discrete metric* ρ *defined by*

$$\rho(x,y) = \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y \end{cases}$$

for $x, y \in X$.

If the cardinality of X *is finite, then an arbitrary dynamical system*

$$(f(x,t))_{x \in X, t \in Z}$$

is periodical.

Proof. Obviously, X is a metric compact space. Let us consider a classical probability measure μ defined in X and being invariant under the group of all motions of $(f(x,t))_{x \in X, t \in \mathbf{R}}$.

Let $X = (x_i)_{1 \leq i \leq n}$. By Theorem 3.11, we obtain the existence of an instant $T_i > 0$ such that $f(x_i, T_i) = x_i$ $(1 \leq i \leq n)$. Let us show that $f(x_i, kT_i) = x_i$ for arbitrary $k \in Z \,\&\, k \geq 0$. Indeed,

$$f(x_i, kT_i) = f((\cdots f(f(x_i, T_i), T_i), \cdots), T_i) = x_i.$$

Analogously, above-mentioned result is valid when $k \in Z \,\&\, k < 0$.

Let T^* be the least common multiple of numbers $(T_i)_{1 \leq i \leq n}$. Clearly,

$$(f(x,t))_{x \in X, t \in Z}$$

is a periodical dynamical system with period T^*. $\square$

We say that a dynamical system defined on a finite set X is homogeneous if a least positive period of the motion of an arbitrary point is equal to its trajectory cardinality.

Remark 3.7 Let $(f(x,t))_{x\in X,t\in Z}$ be a homogeneous dynamical system defined on the finite set X and let $(X_k)_{1\le k\le q}$ be the family of all trajectories. It is clear that the relation $\sum_{1\le k\le q}|X_k| = |X| = n$ holds.

On the one hand, we have

$$\text{l.c.m}(|X_1^*|,\cdots,|X_q^*|) \le \prod_{1\le k\le q}|X_k|,$$

where l.c.m.$(|X_1^*|,\cdots,|X_q^*|)$ denotes the least common multiple of numbers

$$|X_1^*|,\cdots,|X_q^*|.$$

On the other hand,

$$\prod_{1\le k\le q}|X_k| \le \Big(\frac{\sum\limits_{1\le k\le q}|X_k|}{q}\Big)^q = \Big(\frac{n}{q}\Big)^q.$$

Obviously,

$$\max_{1\le q\le n}\Big(\frac{n}{q}\Big)^q \le \exp\Big(\frac{n}{\exp}\Big).$$

Eventually, we arrive at the estimate

$$T^* \le \exp\Big(\frac{n}{\exp}\Big).$$

Let card$(X) = n$. A positive number $T^*(n)$ is called a period of the family of all homogeneous dynamical systems defined on X if $T^*(n)$ is a least positive natural number such that every homogeneous dynamical system $(f(x,t))_{x\in X,t\in Z}$ satisfies the condition $f(x,t+T^*(n)) = f(x,t)$ for every $x\in X$ and $t\in Z$.

The following assertion is valid.

Theorem 3.14 *Let card$(X) = n$. Let $(p_k)_{1\le k\le q}$ be the family of all simple natural numbers such that*

$$(\forall k)(1\le k\le q \to p_k \le n).$$

Let us denote by n_k $(1\le k\le q)$ a largest natural number such that

$$p_k^{n_k} \le n.$$

Then the following formula

$$T^*(n) = \prod_{k=1}^{q} p_k^{n_k}$$

is valid.

Example 3.6 If one assumes the existence of a family $X = \{x_k\}_{1\le k\le n}$ of symbols (letters) such that for some natural number m every word, being in profit, can be represented as the concrete sequence $(y_i)_{1\le i\le m}$, where $y_i\in X$ for arbitrary i, then the process of development of the native language can be considered as the system $(f(x,t))_{x\in X,t\in Z}$. The step of time is the mean time which defines the mean leaving time of the native language. The word transits from one form x to another form $f(x,t)$ after time t. It is also assumed that

the trajectory of a word x is, as usual, the set of all words (so-called synonyms) $(f(x,t))_{t\in Z}$ which represent only one sense during all time of motion. For example, if x corresponds to the word "table" at the instant $t = 0$, then $f(x,t)$ also corresponds to the word "table" at an arbitrary instant t.

Assuming that the system $f(x,t)_{x\in X, t\in Z}$ is dynamical, by Theorem 3.13 it can be easily concluded that there exists a natural number T^* such that the above system is periodical with period T^*.

This quotation means that each dead language will be a living language after concrete time T^* as well each living language will be a dead language after the same time.

If we consider theorems 3.13 and 3.14 for an analogous genetic model of beings, then in our assumptions each creature is 'repeated' (in the sense of genetic composition) after the concrete interval of time T^{**} (by the unit of time is denoted the mean living time of the family).

Example 3.7 (Zermelo's paradox) By Poincare's theorem, the interesting property of recurrence is generic for many dynamical systems, including the motions of the molecules of gas in an insulated bottle (assuming the validity of classical physics), which has a state space of huge dimensionality, say 1025 or much more. One of the most basic and familiar physical facts is that when left alone, macroscopic matter tends to thermodynamic equilibrium, and this happens largely independently of initial conditions or even of the composition of the matter. In the early days of the kinetic theory of gases, J. Clark Maxwell and L.Boltzmann hoped to explain this striking phenomenon by appealing to recurrence. Boltzmann enunciated the ergodic hypothesis, the original form of which posits that the trajectory of any given every initial condition passes through every point of the surface in phase space having the same total energy as the initial condition. This portion of phase space is known as the energy shell, and because of conservation of energy, we may treat the energy shell as the state space. If the ergodic hypothesis were true, Boltzmann felt, then the time averages of physical observables would tend, as $t \to \infty$, to a spatial average, the average of the observable over the energy shell. That average is what we would recognize as the thermodynamic equilibrium for the observable. (Caution: Equilibrium has a different meaning here. Thermodynamic equilibrium certainly does not mean that molecules are at rest, which would only happen at absolute zero temperature.) Enrenfest later realized that the original form of the ergodic hypothesis was probably impossible for physical systems, and modified it to something more in the spirit of Poincare recurrence.

While initially attractive, the ergodic hypothesis leads to some troubling paradoxes, notably Zermelo's paradox: If the gas is initially released from one corner of an insulated bottle, Poincare's recurrence theorem predicts that, according to classical physics, at some time all the molecules will suddenly rush back into the corner of the bottle. According to the ergodic hypothesis, some analogous phenomenon should eventually happen for almost every trajectory. But this has never happened in all of history! The difficulty is that the time required for recurrence in a typical dynamical system of more than very low dimensionality is vast. Following [Thirring, 1980], imagine ten clocks running at a frequency of roughly 1 Hertz, i.e., 1 revolution per second, but suppose that the clocks are not quite accurate, and the precise frequencies are incommensurate (have irrational ratios). This dynamical system is mathematically identical to the free motion on a ten-dimensional torus, with irrational speeds, and every trajectory is recurrent and has an equidistribution property. Let us esti-

mate the time it takes for all the clocks to return to the initial position in synchronism within 1 per cent of a second. The state space can be divided into 1 per cent of 10 is equal to 1020 cells of this margin of error, so it should take 1020 seconds to sample them all and return. The universe is believed to be less than 1018 seconds old.

Example 3.8 Let us show that the state of the nine planets of the solar system is stable in the sense of Poisson.

Indeed, it can be assumed without loss of generality that each planet is a point with mass 1 and moves in its orbit around the Sun not influenced by the motions of the other planets. Then the phase space of such a system is a 27-dimensional space in $\mathbf{R}^{27}$, because every planet has three spatial coordinates. Since the k-th planet moves along the circle, its spatial coordinates $(x_{k1}, x_{k2}, x_{k3})(1 \leq k \leq 9)$ satisfy the condition $x_{k1}^2 + x_{k2}^2 + x_{k3}^2 = r_k^2$, where r_k is the radius of the k-th planet's orbit. This means that the configuration point x representing the state of all nine planets moves along the boundary of the 9-dimensional torus,

$$T^9 = \prod_{i=1}^{9} S_i,$$

where, for arbitrary i $(1 \leq i \leq 9)$ the circle with radius r_i is denoted by S_i.

Let G_k $(1 \leq k \leq 9)$ be the group of all rotations of the circle S_k, and let λ_k be the Haar probability measure defined on $B(S_k)$. It is clear that, on the one hand, $\prod_{k=1}^{n} \lambda_k$ is the $\prod_{k=1}^{n} G_k$-invariant probability Borel measure defined on $\prod_{k=1}^{n} S_i$. On the other hand, the joint motion of the planet system can be described by the formula

$$f(x,t) = (g_{\omega_1}(x_1,t), \cdots, g_{\omega_9}(x_9,t)),$$

where $g_{\omega_k}(x_k,t)$ is the uniform motion of the k-th planet along the circle S_k with a constant angular velocity ω_k. It is clear that, for arbitrary $t \in \mathbf{R}$ and $x \in \prod_{k=1}^{n} G_k$, the value $f(x,t)$ is an element of the group $\prod_{k=1}^{n} G_k$ and, using theorems 3.11 and 3.12, we easily conclude that the joint motion of the solar planet system is stable in the sense of Poisson.

Remark 3.8 It is interesting to observe that as far back as 1734, Daniel Bernoulli considered the plane of orbits of the planets known at the time as accidental points on the sphere surface and showed that they are uniformly distributed. We must say that the theory of uniform distributions is nowadays well developed (see, for example, [106]).

Remark 3.9 The trajectory of each planet of the solar planet system is an ellipse with the Sun put at one of its focuses. Nevertheless, theorems 3.11 and 3.12 hold for such a model of joint motion of the solar system planets.

Remark 3.10 Let Y_0 be a Borel subset of S^2. Then, applying strong versions of theorems 3.11 and 3.12(cf.[46]), ν-almost every points $\Psi^0 \in Y_0$ return to Y_0 under a motion, generated by one-parameter group of transformation $(e^{tA})_{t \in R}$, where A is a normal Hermitian operator in W^2(cf. Theorem 3.10),i.e.,ν-almost every points $\Psi^0 \in Y_0$ are stable in the sense of Poisson.

In context of Remark 3.10, we have more strong results.

Theorem 3.15 *Let A be a normal Hermitian operator in W^2. Then every point in S^2 is stable in the sense of Poisson under a motion, generated by the one-parameter group of transformations $(e^{itA})_{t\in\mathbf{R}}$, where A is a normal Hermitian operator in W^2.*

Proof. Let $\Psi^0 \in S^2$ and let $\varepsilon > 0$. Denote by n_ε a natural number such that

$$\sum_{k>n_\varepsilon} |\Psi_k^0 - \Psi_k^*|^2 < \frac{\varepsilon^2}{2},$$

where $(\Psi_k^0)_{k\in N}$ and $(\Psi_k^*)_{k\in N}$ denote coordinates of Ψ^0 and its diametral point Ψ^* in orthonormal basis of W^2 generated by proper vectors of A, respectively.

Set $G_k = \{e^{it\lambda_k}(\Psi_k^0) : t \in \mathbf{R}\}$, where λ_k is k-th proper number of the operator A.

Without loss of generality, we can assume that $\lambda_k \neq 0$ for $k \in N$.

We can identify G_k with a compact group of rotations of the two-dimensional plane $\mathbf{R}^2$ about its origin.

Let λ_{n_ε} be the Haar measure defined on the compact group $\prod_{1\le k\le n_\varepsilon} G_k$. Applying Poincare's theorem, for ε we can find a such instant $t_0 \in \mathbf{R}^+$ that

$$\sum_{k=1}^{n_\varepsilon} |e^{it_0\lambda_k}(\Psi_k^0) - \Psi_k^0|^2 < \frac{\varepsilon^2}{2}.$$

Finally, we get

$$||e^{it_0A}(\Psi^0) - \Psi^0||_1^2 \le \sum_{k=1}^{n_\varepsilon} |e^{it_0\lambda_k}(\Psi_k^0) - \Psi_k^0|^2 +$$

$$\sum_{k>n_\varepsilon} \left|\Psi_k^0 - \Psi_k^*\right|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.$$

This ends the proof of Theorem 3.15. □

Remark 3.11 In some situations, when a metric space X is equipped with a σ-finite invariant measure, one cannot deduce that μ-almost every trajectory is stable in the sense of Poisson. Indeed, let us consider the system of differential equations

$$\frac{dx_i}{dt} = 1, \frac{dx_2}{dt} = 0, \cdots, \frac{dx_n}{dt} = 0,$$

whose solution has the form

$$x_1 = x_1^{(0)} + t, x_2 = x_2^{(0)}, \cdots, x_n = x_n^{(0)}$$

and tends to infinity as $t \to +\infty$ or $t \to -\infty$. It is clear that the solution is not stable in the sense of Poisson.

We say that a point y is an ω-limit point for motion $(f(x,t))_{t\in\mathbf{R}}$ $(x \in X)$ if, for an arbitrary parametric sequence $(t_k)_{k\in N}$ tending to $+\infty$, the point y is a limit point for the sequence $(f(x,t_k))_{k\in N}$.

Analogously, a point $z \in X$ is called an α-limit point for the motion $(f(x,t))_{t\in\mathbf{R}}$ if, for an arbitrary parameter sequence $(t_k)_{k\in N}$ tending to $-\infty$, the point z is a limit point for the sequence $(f(x,t_k))_{k\in N}$.

We say that a point $x \in X$ tends to ∞ as $t \to +\infty$ if the trajectory $(f(x,t))_{t\in\mathbf{R}}$ has no ω-limit points.

Analogously, a point $y \in X$ is called tending to ∞ as $t \to -\infty$ if the trajectory $(f(y,t))_{t\in\mathbf{R}}$ has no α-limit points.

The following proposition is a generalization of the Poincaré- Carathéodory theorem.

Theorem 3.16 (E.Hopf) *Let X be a locally compact separable metric space, and let $(f(x,t))_{x\in X,t\in\mathbf{R}}$ be a dynamical system with an invariant σ-finite Radon measure μ. Then μ-almost every point of X is stable in the sense of Poisson or tends to infinity as $t \to \infty$.*

The proof of Theorem 3.16 can be found e.g. in [120].

The following important statement was obtained by Birkhoff and Chintchin (see [48]).

Theorem 3.17 *Let (E,S,μ) be a measurable space with probability measure and g be some measurable transformation of E. If the measure μ is invariant under the transformation g, then for an arbitrary μ-integrable function f we have*

$$a)\ \ \mu(\{x\ :\ (\exists \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(g^k x) = f^*(x)\}) = 1,$$

where the function f^* is μ-integrable and μ-almost invariant under the transformation g, i.e.

$$\mu(\{x\ : x \in E\ \&\ f^*(g(x)) = f^*(x)\}) = 1,$$

$$b)\ \ \int_E f^*(x)d\mu(x) = \int_E f(x)d\mu(x).$$

Now, let I be a nonempty countable set of parameters.

Definition 3.1 The family $\{g^t\}_{t\in\mathbf{R}}$ of transformations of $\mathbf{R}^I$ is called a one-parameter group if

$$(\forall s)(\forall t)(s \in \mathbf{R}\ \&\ t \in \mathbf{R} \to g^{t+s} = g^t \cdot g^s)$$

and g^0 is the identity transformation of the space $\mathbf{R}^I$.

Remark 3.12 Note that every dynamical system (in the sense of Birkhoff) $(f(x,t))_{x\in X,t\in\mathbf{R}}$, defined in the phase space X, is one-parameter group of transformations of X.

Definition 3.2 $(\mathbf{R}^I, \{g^t\}_{t\in\mathbf{R}})$ is called a phase flow if $\{g^t\}_{t\in\mathbf{R}}$ is a one-parameter group of transformations of the space $\mathbf{R}^I$.

Definition 3.3 A transformation $g = \{g^t\}_{t\in\mathbf{R}} : \mathbf{R}\times\mathbf{R}^I \to \mathbf{R}^I$ is called one-parameter group of diffeomorphisms of $\mathbf{R}^I$ if the following four conditions are fulfilled:

1) $g(t,x) = g^t x$ for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^I$;
2) g is a differentiable transformation on $\mathbf{R}\times\mathbf{R}^I$;
3) $g^t : \mathbf{R}^I \to \mathbf{R}^I$ is a diffeomorphism for $t \in \mathbf{R}$;

4) the family $\{g^t\}_{t\in\mathbf{R}}$ is an one-parameter group of transformations of $\mathbf{R}^I$.

Let $(\mathbf{R}^I, \{g^t\}_{t\in R})$ be the phase flow defined by the one-parameter group of diffeomorphisms of $\mathbf{R}^I$.

Definition 3.4 The velocity of a phase point defined by

$$v(x) = \frac{d}{dt} g^t(x)|_{t=0}$$

is called a phase velocity $v(x)$ of the phase flow g^t at the point $x \in \mathbf{R}^I$.

Definition 3.5 Let μ be a measure defined on some $\{g^t\}_{t\in\mathbf{R}}$-invariant σ-algebra of subsets of the space $\mathbf{R}^I$. We say that the phase flow $(\mathbf{R}^I, \{g^t\}_{t\in\mathbf{R}})$ preserves the measure μ if

$$(\forall t)(\forall B)(t \in \mathbf{R} \ \& \ B \in \mathrm{dom}(\mu) \to \mu(g^t(B)) = \mu(B)).$$

Theorem 3.18 *Let $A = (a_{ij})_{i,j\in\overline{1,n}}$ be an n-dimensional matrix with $|det(A)| = 1$. Then*

$$(\forall B)(B \in B(R^n) \to \ell_n(A(B)) = \ell_n(B)),$$

where $B(\mathbf{R}^n)$ denotes the Borel σ-algebra of subsets of $\mathbf{R}^n$ and ℓ_n denotes the n-dimensional Lebesgue measure defined in $\mathbf{R}^n$.

The proof of Theorem 3.18 can be obtained by using the formula of changing of variables in Lebesgue integral.

Let $\mu = \nu_\Delta$, where ν_Δ is the measure constructed in Chapter 5 for $\Delta = [0;1[^N$.
The following result is valid.

Theorem 3.19 *Let $\overline{D_n}$ be the group of all n-dimensional $A = (a_{ij})_{i,j\in\overline{1,n}}$ matrices with $|det(A)| = 1$. Let us denote*

$$G = \cup_{n\in N}(\overline{D_n} \times I^{(n)}),$$

where $I^{(n)}$ denotes the identity transformation of the topological vector space $\mathbf{R}^{N\setminus\{1,\cdots,n\}}$. Then the measure μ is G-invariant.

Proof. Note that, for an arbitrary natural number n, the measure μ can be considered as the product of two measures μ_1 and μ_2, where μ_1 coincides with the n-dimensional Lebesgue measure. The measure μ_1 is $\overline{D_n}$-invariant (see Theorem 3.18). The measure μ_2 can be considered as an $I^{(n)}$-invariant measure. By using the well-known result on the product of σ-finite invariant Borel measures, we conclude that the measure $\mu = \mu_1 \times \mu_2$ is $\overline{D_n} \times I^{(n)}$-invariant and Theorem 3.19 is proved. $\square$

Theorem 3.20 *Let $A = (a_{ij})_{i,j\in\overline{1,n}}$ be an n-dimensional matrix. Then*

$$det(e^{t\cdot A}) = e^{t\cdot Tr(A)},$$

where $Tr(A) = \sum_{i=1}^{n} a_{ii}$.

Definition 3.6 We say that there exists a vector field A defined on $\mathbf{R}^I$ if the value of the vector quantity A is specified at each point x of $\mathbf{R}^I$, i.e., $A = A(x)$. It is clear that an arbitrary

family of transformations $\{g^t\}_{t\in\mathbf{R}}$ of $\mathbf{R}^I$ can be considered as a vector field defined on the corresponding vector space. If $\{g^t\}_{t\in\mathbf{R}}$ is the family of differentiable transformations, then $(\mathbf{R}^I, \{g^t\}_{t\in\mathbf{R}})$ is called a vector flow on $\mathbf{R}^I$.

We will deal with a stationary field which does not change as time passes. If such a variation takes place, we will consider the field at a fixed moment of time and thus reduce our consideration to a stationary field. As examples of vector fields, we can consider a field of velocities v, a field of momentum density $\rho\mathbf{v}$ (where ρ is the mass distribution density) for a liquid or gas flow, a field of force F, an electric field E (where E is electric field strength), etc.

Definition 3.7 The vector field defined by

$$(\forall x)(x \in \mathbf{R}^I \to v(x) = \frac{d}{dt} g^t(x)|_{t=0})$$

is called a vector field of velocities v on $\mathbf{R}^I$ determined by the phase flow $(\mathbf{R}^I, \{g^t\}_{t\in\mathbf{R}})$.

Definition 3.8 We say that a vector field of velocities v defined on $\mathbf{R}^I$ preserves the measure μ with $\mathrm{dom}(\mu) = B(\mathbf{R}^I)$ if

$$(\forall B)(\forall t)(B \in \mathrm{dom}(\mu)\ \&\ t \in \mathbf{R} \to g^t(B) \in \mathrm{dom}(\mu)\ \&\ \mu(g^t(B)) = \mu(B)).$$

Definition 3.9 The divergence div$v$ of the vector field of velocities v on $\mathbf{R}^I$ is defined by

$$\mathrm{div} v = \sum_{i\in I} \frac{\partial v_i}{\partial x_i},$$

where $(v_i)_{i\in I}$ is the family of components of the vector velocity v and $x_i \in \mathbf{R}$ for all $i \in I$.

Theorem 3.21(Liouville) *Let $v = A(x)$ be a linear continuously differentiable vector field of velocities defined on $\mathbf{R}^n$, and $D(0)$ be some Borel subset of $\mathbf{R}^n$. Then the formula*

$$\frac{d\ell_n(D_t)}{dt} = \int_{D(t)} div(A) d\ell_n$$

is valid, where $D(t)$ denotes the state of the subset $D(0)$ at the moment t under the action of the phase flow defined by the vector field of velocities $v = A(x)$.

The proof of Theorem 3.21 can be found in [1].

Remark 3.13 Let H be a twice continuously differentiable function of $2n$ variables $p_1, \cdots, p_n, q_1, \cdots, q_n$. A system of $2n$ variables defined by

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},\ \dot{q}_i = \frac{\partial H}{\partial p_i},\ i = 1, \cdots, n \qquad (3.1)$$

is called a canonical system of Hamilton's equations. Hamilton had showed that various models in mechanics, electrostatics, optics, variation calculus and in other regions of applications can be described by (3.1)(cf.[1],p.71). As

$$\mathrm{div}\mathbf{v} = \sum_{k=1}^{n} (\frac{\partial^2 H}{\partial p_k \partial q_k} - \frac{\partial^2 H}{\partial q_k \partial p_k}) = 0,$$

using Theorem 3.21, we easily establish that vector flow of phase velocities defined by the canonical system of Hamilton's equations preserves the $2n$-dimensional Lebesgue measure. This law is known in the literature as the law of preservation of energy.

In the context of Remark 3.13 we present the following example from electrostatics.

Example 3.9 The force acting between the constituents (electrons and nuclei) of matter is given by Coulomb's inverse square law of electrostatics: if two particles have charges q_1 and q_2 and locations x_1 and x_2 in $\mathbf{R}^3$, then F_1-the force on the first due to the second–minus F_2 - the force on the second due to the first, and is given by

$$-F_2 = F_1 = q_1 q_2 \frac{(x_1 - x_2)}{|x_1 - x_2|^3}.$$

If $q_1 q_2 < 0$, then the force is attractive; otherwise, it is repulsive. This force can be written as minus the gradient (denoted by ∇) of a potential energy function

$$W(x_1, x_2) = q_1 q_2 \frac{1}{|x_1 - x_2|},$$

i.e.,

$$F_1 = -\nabla_1 W \text{ and } F_2 = -\nabla_2 W.$$

If there are N electrons located at $\overline{X} = (x_1, \cdots, x_N)$ with $x_i \in \mathbf{R}^3$, and k nuclei with positive charges $\overline{Z} = (z_1, \cdots, z_k)$ and located at $\overline{\mathbf{R}} = (\mathbf{R}_1, \cdots, \mathbf{R}_k)$ with $\mathbf{R}_i \in \mathbf{R}^3$, then the total-potential energy function is

$$W(\overline{X}) = -A(\overline{X}) + B(\overline{X}) + U$$

with

$$A(\overline{X}) = e^2 \sum_{i=1}^{N} V(x_i),$$

$$V(x) = \sum_{j=1}^{k} z_j |x - \mathbf{R}_j|^{-1},$$

$$B(\overline{X}) = e^2 \sum_{1 \le i < j \le N} |x_i - x_j|^{-1},$$

$$U = e^2 \sum_{1 \le i < j \le k} z_i z_j |\mathbf{R}_i - \mathbf{R}_j|^{-1}.$$

The A term is the electron-nucleus attractive potential energy with $eV(x)$ being the electric potential of the nuclei. B is the electron-electron repulsive energy and U is the repulsive energy of the nuclei. A, B, U and V depend on $\overline{R}$ and $\overline{Z}$, which are fixed and, therefore, do not appear explicitly in the notation. It is then the case that the force on the i-th particle is

$$F_i = -\nabla_i W.$$

In the case of an atom, $k = 1$, by definition. The case $k > 1$ is called a molecular case, but it includes not only molecules of very small sizes but also solids, which are really only huge molecules.

So far, this has been just classical electrostatics and we now turn our attention to classical dynamics. Newton's law of motion is (with a dot denoting $\frac{d}{dt}$, where t is the time)

$$m\,\ddot{\mathrm{x}}_i = F_i.$$

This law of motion, which is a system of second order differential equations, is equivalent to the following system of first order equations. We introduce the Hamiltonian function which is the function on the phase space $\mathbf{R}^{6N} = (\mathbf{R}^3 \times \mathbf{R}^3)^N$ given by

$$H(\overline{P},\overline{X}) = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + W(\overline{X}).$$

The notation $\overline{P} = (p_1, \cdots, p_N)$ with p_i in $\mathbf{R}^3$ is used, and the quantity

$$T = \frac{1}{2m}\sum_{i=1}^{N} p_i^2$$

is called a kinetic energy. The above equations of motion are equivalent to the following first order system in $\mathbf{R}^{6N}$:

$$v_i \equiv \dot{\mathrm{x}}_i = \frac{\partial H}{\partial p_i},$$

$$\dot{\mathrm{p}}_i = -\frac{\partial H}{\partial x_i}.$$

The velocity of the i-th electron is v_i, and p_i is called its momentum: $p_i = mv_i$ by the first equation above.

According to the well-known Schwartz' theorem, we have

$$\frac{\partial^2 H}{\partial x_j \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial x_j}$$

and, hence

$$\mathrm{div}(V(\overline{X},\overline{P})) = \sum_{i=1}^{N}\left(\frac{\partial^2 H}{\partial x_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial x_i}\right) = 0.$$

By Liouville's theorem, the vector flow defined by the vector field of velocities

$$\left(\frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_N}, -\frac{\partial H}{\partial x_1}, \cdots, -\frac{\partial H}{\partial x_N}\right)$$

preserves the Lebesgue measure $dx_1 \cdots dx_N dp_1 \cdots dp_N$ on $\mathbf{R}^{6N}$.

Theorem 3.22 *Let us consider the differential equation*

$$\frac{d\Psi}{dt} = A\Psi, \tag{3.2}$$

where

$$A = \begin{pmatrix} A_n\ 0 \cdots & \\ 0\ 0 & \\ \vdots & \ddots \end{pmatrix},$$

A_n is an n-dimensional matrix with $Tr(A_n) = 0,\ t \in \mathbf{R}$ and $\Psi \in R^N$. Then the phase flow $(\mathbf{R}^N, \{g^t(\cdot)\}_{t\in\mathbf{R}}) = (\mathbf{R}^N, \{e^{tA}(\cdot)\}_{t\in\mathbf{R}})$ defined by the vector field of velocities (3.2) preserves the measure μ.

Proof. It is clear that $e^{tA} \times \Psi_0$ is a solution of equation (3.2) with the initial condition $\Psi(0) = \Psi_0$. On the one hand, by Theorem 3.20,

$$\det(e^{tA_n}) = e^{tTr(A_n)} = 1.$$

On the other hand,

$$e^{tA} = e^{tA_n} \times I^{(n)} \in \overline{D_n} \times I^{(n)} \subset G.$$

Theorem 3.21 implies that the phase flow defined by (3.2) preserves the measure μ.

The proof is completed. $\square$

Theorem 3.23 *Let us consider the differential equation*

$$\frac{d\Psi}{dt} = b, \tag{3.3}$$

where $\Psi \in \mathbf{R}^N,\ t \in \mathbf{R}$ and $b = (b_1, b_2, \cdots) \in \mathbf{R}^N$.

Then the phase flow $(\mathbf{R}^N, \{b \times t + (\cdot)\}_{t\in\mathbf{R}})$ defined by the vector field of velocities (3.3) preserves the measure μ if and only if $b \in \ell_1$.

Remark 3.14 The proof of Theorem 3.23 can be obtained obviously by using Theorem 5.1.

Theorem 3.24 *Let us consider the differential equation*

$$\frac{d\Psi}{dt} = A\Psi, \tag{3.4}$$

where $\Psi \in R^N,\ t \in R$ and A is an infinite-dimensional diagonal matrix.

Then the phase flow defined by the vector field of velocities (3.4)(being one-parameter group of transformations of $\mathbf{R}^N$) preserves the measure μ if and only if the series $Tr(A)$ is absolutely convergent and $Tr(A) = 0$.

Proof. Necessity. Let the phase flow defined by (3.4) preserve the measure μ. Let us show that $Tr(A) = 0$. Indeed, if we assume that $Tr(A) \neq 0$, then $e^{tTr(A)} \neq 0$, too. It means that

$$\mu(e^{tA}([0;1[^N)) = \mu(\prod_{i\in N}[0; e^{ta_i}[),$$

where $(a_i)_{i\in N}$ is a sequence of all diagonal elements of the matrix A.

From the e^{At}-invariance of the measure μ, we have

$$\lim_{n\to\infty} \prod_{k\geq n} \lambda_k([0; e^{a_k}[) = 1,$$

where λ_k denotes the normed Lebesgue measure defined in $[0;1[$.

On the other hand, we have

$$\lim_{n\to\infty} \prod_{1\leq k\leq n} \lambda_k([0; e^{ta_k}[) = e^{\lim_{n\to\infty}\sum_{k=1}^{n} ta_k} = e^{tTr(A)} \neq 1,$$

where μ_k denotes the Lebesgue measure defined in $\mathbf{R}$, and we get a contradiction with the e^{tA}-invariance of the measure μ.

Now, assume that $Tr(A)$ is not absolutely convergent. Let $(a^+_{k_n})_{n\in N}$ and $(a^-_{k_p})_{p\in N}$ respectively denote the subsequences of all positive and negative members of the sequence $(a_k)_{k\in N}$. It is clear that

$$\sum_{n\in N} a^+_{k_n} = +\infty \ \& \ \sum_{p\in N} a^-_{k_p} = -\infty.$$

On the one hand, we have

$$\lim_{n\to\infty}\prod_{k\geq n}\lambda_k(\prod_{k\geq n}[0;1[_k \cap \prod_{k\geq n} e^{ta_k}\cdot[0;1[_k) = 1.$$

On the other hand, for $t>0$, we have

$$\lim_{n\to\infty}\prod_{k\geq n}\lambda_k(\prod_{k\geq n}[0;1[_k \cap \prod_{k\geq n}[0;e^{ta_k}[) =$$

$$= \lim_{n\to\infty}\prod_{p\geq n} e^{ta_{k_p}} = e^{\lim_{n\to\infty}\sum_{p\geq n} ta_{k_p}} = e^{-\infty} = 0.$$

The necessity is proved.

Sufficiency. Let $Tr(A)=0$ and the series $\sum_{k\geq 1} a_k$ be absolutely convergent. It is sufficient to show that

$$\mu(e^{tA}(B\times[0;1[^{N\setminus\{1,\cdots,n\}}) = \mu(B\times[0;1[^{N\setminus\{1,\cdots,n\}}),$$

where $B\in B(R^n)$ $(n\in N)$.

Indeed, on the one hand, we have

$$\lim_{n\to\infty}\prod_{k\geq n}\lambda_k(\prod_{k\geq n}[0;e^{ta_k}[) =$$

$$\lim_{n\to\infty}\prod_{p\geq n}\lambda_{k_p}(\prod_{p\geq n}[0;e^{ta^-_{k_p}}[) = e^{\lim_{n\to\infty}\sum_{p\geq n} ta^-_{k_p}} = e^0 = 1.$$

On the other hand, we have

$$\mu(e^{tA}(B\times\prod_{k\geq n}[0;1[_k)) = \lim_{m\to\infty}((\prod_{1\leq i\leq m}\mu_i)(e^{tA_m}(B\times\prod_{n+1\leq i\leq m}[0;1[_i))\times$$

$$\times\prod_{i>m}\lambda_i(\prod_{i>m}[0;e^{ta_i}[) =$$

$$\lim_{m\to\infty} e^{\sum_{1\leq i\leq m} ta_i}\times(\prod_{1\leq i\leq n}\mu_i)(B)\times\lim_{m\to\infty}(\prod_{i\geq m+1}\lambda_i)(\prod_{i\geq n+1}[0;e^{ta_i}[) =$$

$$= e^{\sum_{i\geq 1} ta_i}\times(\prod_{1\leq i\leq n}\mu_i)(B) = \mu(B\times\prod_{k>n}[0;1[_k),$$

where A_m is the m-dimensional matrix standing in the upper-left part of the matrix A.

The sufficiency is proved. $\square$

Remark 3.15 If A is an infinite-dimensional diagonal matrix and $Tr(A)$ is absolutely convergent, then, using the method considered above, we obtain

$$(\forall B)(B \in B(\mathbf{R}^N) \to \mu(e^{tA}(B)) = e^{tTr(A)} \times \mu(B)).$$

Thus, we get the validity of the following result which is a direct analog of Liouville's theorem (see Theorem 3.21.)

Theorem 3.25 *Let $v = Ax$ be a vector field of velocities defined on $\mathbf{R}^N$, where A is an infinite-dimensional diagonal matrix such that $Tr\,(A)$ is absolutely convergent. Let $D(0)$ be some Borel subset in $\mathbf{R}^N$. Then the formula*

$$\frac{d\mu(D(t))}{dt} = \int\limits_{D(t)} div(A)d\mu$$

is valid, where $D(t)$ denotes the state of the subset $D(0)$ at the moment t under the action of the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$.

Theorem 3.26 *Let us consider the differential equation*

$$\frac{d\Psi}{dt} = A\Psi + f\ , \tag{3.5}$$

where $\Psi \in \mathbf{R}^N$, f is a continuous vector-function in $\mathbf{R}^N$ (of a parameter t) and, A is an infinite-dimensional diagonal matrix with absolutely convergent trace such that $Tr\,(A) = 0$. Then the phase flow defined by (3.5) preserves the measure μ if and only if the following condition

$$(\forall t)(t > 0 \to \int_0^t e^{(t-\tau)A} f(\tau)d\tau \in \ell_1)$$

holds.

Proof. It is clear that a solution of (3.5) with initial condition $\Psi(0) = \Psi_0$ has the following form:

$$\Psi(t) = e^{At}\Psi_0 + \int_0^t e^{(t-\tau)A} f(\tau)d\tau.$$

Hence, a motion $(g^t)_{t\geq 0}$ defined by (3.5) has the following form

$$(\forall \Psi)(\Psi \in \mathbf{R}^N \to g^t(\Psi) = e^{At}\Psi + \int_0^t e^{(t-\tau)A} f(\tau)d\tau 1).$$

Necessity. Let the phase flow $(g^t)_{t\geq 0}$ defined by (3.5) preserves the measure μ, i.e.,

$$(\forall X)(X \in B(\mathbf{R}^N) \Rightarrow \mu(g^t(X)) = \mu(e^{At}(X) + \int_0^t e^{(t-\tau)A} f(\tau)d\tau = \mu(X)).$$

If we put $X = e^{-At}(Y)$, where $Y \in B(\mathbf{R}^N)$, we get

$$(\forall Y)(Y \in B(\mathbf{R}^N) \to \mu(Y) = \mu(Y + \int_0^t e^{(t-\tau)A} f(\tau)d\tau)).$$

Using Theorem 5.1, we conclude that $(\forall t)(t \geq 0 \to \int_0^t e^{(t-\tau)A} f(\tau)d\tau \in \ell_1)$ and the necessity is proved.

Sufficiency. The sufficiency obviously follows from the theorems 3.24 and 5.1. This ends the proof of Theorem 3.26. □

Remark 3.15 Note that the Green's matrix $h_+ = e^{(t-\tau)A}$ is a reaction on the collection of asymmetrical impulses $(f_i(t))_{i\in N} = \delta_+(t)$ $(t \in \mathbf{R})$. Let a_{ii} $(i \in N)$ be the family of diagonal elements of the matrix A in Theorem 3.26 and $(\alpha_i)_{i\in N} \in \ell_1$. We put $f_i(t) = \frac{\alpha_i e^{a_{ii}t}}{1+t^2}$ for $i \in N$, $t \in \mathbf{R}$. Then the family $(f_i(t))_{i\in N}$ gives an example of asymmetrical impulses when the phase flow defined by (3.5) preserves the measure μ.

The following auxiliary proposition plays a key role in our further investigations.

Lemma 3.5 *Let A be a bounded by the norm Hermitian operator in infinite-dimensional separable Hilbert space ℓ_2. Then all proper numbers* [3] *of the operator A are real and there exists an orthonormal basis $(\Psi_k)_{k\in N}$ generated by corresponding proper vectors of A.*

The proof of Lemma 3.5 can be found in [94].

In the sequel we consider so-called normal Hermitian operators ℓ_2 which have the following property: there exists an orthonormal basis consisting of its proper vectors.

Let A be a normal Hermitian operator in ℓ_2 with the convergent trace $Tr(A)$. Let $(\Psi_k)_{k\in N}$ be an orthonormal basis in ℓ_2 generated by proper vectors of the operator A. Let $B : \mathbf{R}^N \to \ell_2$ be a linear operator defined by $B(e_k) = \frac{1}{k}\Psi_k$ for $k \in N$, where $(e_k)_{k\in N}$ denotes a standard basis in ℓ_2. We set

$$(\forall X)(X \in B(\ell_2) \Rightarrow \mu_A(X) = \mu(B^{-1}(X))), \quad (3.6)$$

where $B(\ell_2)$ denotes a Borel σ-algebra of subsets of ℓ_2.

The following assertions are valid.

Theorem 3.27 *Let a vector field of phase velocities in ℓ_2 be defined by* (3.4), *where $\Psi \in \ell_2$ and A is a normal Hermitian operator in ℓ_2 with a convergent trace $Tr(A)$. Then the phase flow defined by* (3.4) *preserves the measure μ_A defined by* (3.6) *if and only if $Tr(A) = 0$ and $Tr(A)$ is absolutely convergent.*

Theorem 3.28 *Let A be a normal Hermitian operator with an absolutely convergent trace $Tr(A)$ in ℓ_2. Let $D(0)$ be some Borel subset in ℓ_2. Then the formula*

$$\frac{d\mu_A(D(t))}{dt} = \int_{D(t)} div(A)d\mu_A \qquad (3.7)$$

is valid, where μ_A is defined by (3.6), *$D(t)$ denotes the state of the subset $D(0)$ at the moment t $(t \in \mathbf{R})$ under the action of the phase flow defined by* (3.4).

In addition, let us remark that one can obtain the validity of an analog of theorem 3.26 in the space ℓ_2.

Let us consider the following notions characterizing the behavior of various dynamical systems defined by vector field of velocities $v = A \times \Psi$ in $\mathbf{R}^N$, where A is an infinite-dimensional real-valued diagonal matrix.

[3] A number $\lambda \in \mathbf{C}$ is called a proper number for a linear operator $A : \ell_2 \to \ell_2$ if there exists a non-zero element $u \in \ell_2$ such that $Au = \lambda u$. The element u is called a proper vector of A corresponding to the proper number λ.

Definition 3.10 We say that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is stable in the sense of a Borel measure ν defined in $\mathbf{R}^N$ if it preserves the measure ν.

Definition 3.11 We say that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is pressing in the sense of a Borel measure ν if

$$(\forall t_1)(\forall t_2)(\forall D)(0 < t_1 < t_2 < \infty \;\&\; 0 < \nu(D) < \infty \rightarrow \nu(e^{t_1 A}(D)) < \nu(e^{t_2 A}(D))).$$

Definition 3.12 We say that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is totally pressing in the sense of a Borel measure ν if

$$(\forall t)(\forall D)(0 < t < \infty \;\&\; 0 < \nu(D) < \infty \rightarrow \nu(e^{tA}(D)) = 0).$$

Definition 3.13 We say that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is expansible in the sense of a Borel measure ν if

$$(\forall t_1)(\forall t_2)(\forall D)(0 < t_1 < t_2 < \infty \;\&\; 0 < \nu(D) < \infty \rightarrow \nu(e^{t_1 A}(D)) < \nu(e^{t_2 A}(D))).$$

Definition 3.14 We say that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is totally expansible in the sense of a Borel measure ν if

$$(\forall t)(\forall D)(0 < t < \infty \;\&\; 0 < \nu(D) < \infty \rightarrow \nu(e^{tA}(D)) = \infty).$$

In context of definitions 3.10-3.14, the following example is of interest.

Example 3.10 (Infinite continuous Malthusian growth model) The differential equation describing the continuous Malthusian growth model says that the derivative of an unknown Population function $P(t)$ is proportional the unknown Population function. The only function that is equal to the derivative of itself is the exponential function. We try to find a solution of the form $P(t) = ce^{rt}$, where c is an arbitrary constant. But $P^{'}(t)$ is $c \times r \times e^{r \times t}$, which is $r \times P(t)$, so satisfies the differential equation. If we add an initial condition $P(0) = P_0$, then the unique solution becomes $P(t) = P_0 \times e^{r \times t}$. This is another reason why Malthusian growth is often called exponential growth.

Now, let us consider an infinite non-antagonistic family of populations and let $\Psi_k(t)$ be the population function of the k-th Population. Then the continuous Malthusian growth model for an infinite family of non-antagonistic populations is described by the following linear differential equation

$$\frac{d\Psi(t)}{dt} = A\Psi(t)$$

with an initial condition $\Psi(0) = \Psi_0 \in (\mathbf{R}^+)^N$, where A is an infinite-dimensional real-valued diagonal matrix, $\Psi(t) = (\Psi_k(t))_{k \in N} \in (\mathbf{R}^+)^N$.

If $\mathrm{div}(A)$ is absolutely convergent, then, applying infinite-dimensional analog of Lioville's theorem, we conclude that the phase flow $(\mathbf{R}^N, e^{tA} \times (\cdot))$ is:

(i) stable in the sense of the measure μ if $\mathrm{div}(A) = 0$;

(ii) pressing in the sense of the measure μ if $\mathrm{div}(A) < 0$;

(iii) expansible in the sense of the measure μ if $\mathrm{div}(A) > 0$.

Let μ be a diffused probability Borel measure defined on the Polish space E and let $B(E)$ be a Borel σ-algebra of subsets of E.

Definition 3.15. A measurable automorphism $g : E \to E$ is called admissible in the sense of quasi-invariance for the measure μ if

$$(\forall X)(X \in B(E) \to (\mu(X) = 0 \Leftrightarrow \mu(g(X)) = 0)).$$

Theorem 3.29 *The class Q_μ of all admissible automorphisms in the sense of quasi-invariance of the measure μ is a group with respect to the usual composition of automorphisms.*

Definition 3.16 A measurable automorphism $g : E \to E$ is called admissible in the sense of invariance for the measure μ if

$$(\forall X)(X \in B(E) \to \mu(X) = \mu(g(X))).$$

Theorem 3.30 *The class I_μ of all admissible automorphisms (in the sense of invariance) of the measure μ is a group with respect to the usual composition of automorphisms.*

Let show that that the notions of dynamical (i.e.,$(E, B(E), I_\mu, \mu)$) and quasi-dynamical (i.e.,$(E, B(E), I_\mu, \mu)$) systems canonically associated with any Borel diffused probability measure μ on E are different. In this direction we have the following

Theorem 3.31

$$I_\mu \subset Q_\mu \ \& \ Q_\mu \setminus I_\mu \neq \emptyset.$$

Proof. Let $g \in I_\mu$. Then

$$(\forall X)(X \in B(E) \to \mu(g(X)) = \mu(X)).$$

In particular, if $\mu(X) = 0$, then $\mu(g(X)) = \mu(X) = 0$.

If $\mu(g(X)) = 0$, then from the invariance of the measure μ we have

$$\mu(X) = \mu(g^{-1}(g(X))) = \mu(g(X)) = 0.$$

We get $g \in Q_\mu$ and an inclusion $I_\mu \subset Q_\mu$ is proved.

Now let show that $Q_\mu \setminus I_\mu \neq \emptyset$.

Let consider a standard Gaussian Borel measure γ on the real axis R. By Lemma 7.1, there exists a Borel isomorphism

$$\Phi : E \to R$$

such that $(\forall X)(X \in B(E) \to \mu(X) = \gamma(\Phi(X)))$.

It is clear that

$$(\forall h)(\forall X)(h \in R \ \& \ X \in B(R) \to (\gamma(X) = 0 \ \Leftrightarrow \ \gamma(X + h) = 0))$$

and there exists no $X_0 \in B(R)$ with $0 < \gamma(X_0) < 1$ that

$$(\forall h)(h \in R \to \gamma(X_0 + h) = \gamma(X_0)).$$

Let consider a group of automorphisms G_0 of E defined by

$$G_0 = \{\Phi \circ \Phi_h \circ \Phi^{-1} : h \in R\},$$

where Φ_h is a shift in R defined by $(\forall x)(x \in R \rightarrow \Phi_h(x) = x + h)$.

Then we have

$$\mu(X) = 0 \Leftrightarrow \gamma(\Phi(X)) = 0 \Leftrightarrow \gamma(\Phi_h(\Phi(X))) = 0 \Leftrightarrow \gamma(\Phi^{-1}(\Phi_h(\Phi(X)))) = 0 \Leftrightarrow$$

$$\mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X)) = 0.$$

The last relation means that $G_0 \subset Q_\mu$.

Let $Y_0 \in B(R)$ and $h_0 \in R$ such that

$$\gamma(Y_0 + h_0) \neq \gamma(Y_0).$$

We set $X_0 = \Phi^{-1}(Y_0)$.

On the one hand, we have

$$\mu(X_0) = \gamma(\Phi(X_0)) = \gamma(\Phi(\Phi^{-1}(Y_0))) = \gamma(Y_0).$$

On the other hand, we have

$$\mu(\Phi \circ \Phi_{h_0} \circ \Phi^{-1}(X_0)) = \gamma(\Phi \circ \Phi_{h_0} \circ \Phi^{-1} \circ \Phi(X_0)) =$$

$$\gamma(\Phi(X_0) + h_0) = \gamma(Y_0 + h_0).$$

Thus, we deduce that $\Phi \circ \Phi_{h_0} \circ \Phi^{-1} \in Q_\mu \setminus I_\mu$.

This ends the proof of Theorem 3.31. $\square$

Theorem 3.32

$$card(Q_\mu) = card(I_\mu) = c,$$

where c denotes the cardinality of the continuum.

Proof. Let us denote by $A(E)$ the group of all Borel automorphisms of E. Since $I_\mu \subseteq Q_\mu \subseteq A(E)$ and $\mathrm{card}(A(E)) \leq c$, we have

$$\mathrm{card}(I_\mu) \leq \mathrm{card}(Q_\mu) \leq \mathrm{card}(A(E)) \leq c.$$

Let S be the unit circle in the Euclidean plane $\mathbb{R}^2$. We may identify the circle S with a compact group of all rotations of $\mathbb{R}^2$ about its origin. Let λ be the probability Haar measure defined on the compact group S. By Lemma 7.1, there exists Borel isomorphisms $\Phi : E \rightarrow S$ such that

$$(\forall X)(X \in B(E) \rightarrow \mu(X) = \lambda(\Phi(X))).$$

Let consider a group G_0 of measurable automorphisms of E defined by

$$G_0 = \{\Phi \circ g \circ \Phi^{-1} : g \in S\}.$$

It is clear that $G_0 \subseteq I_\mu$ and $\mathrm{card}(G_0) = c$.

The last relation means that

$$\mathrm{card}(I_\mu) \geq c.$$

This ends the proof of Theorem 3.32. $\square$

Theorem 3.33.

$$A(E) \setminus Q_\mu \neq \emptyset.$$

Proof. Let Γ be a canonical Gaussian Borel measure defined on R^N. Following Kakutani well-known result(cf. Corollary 4.4), we have

$$(\exists h_0)(\exists X_0)(h_0 \in R^N \setminus \ell_2 \ \& \ X \in B(R^N) \rightarrow \Gamma(X_0) = 0 \ \& \ \Gamma(X_0 + h_0) > 0).$$

Let a mapping : $E \rightarrow R^N$ be a Borel isomorphism between measures μ and Γ.

Let Φ_{h_0} be a shift in R^N defined by $(\forall x)(x \in R^N \rightarrow \Phi_{h_0}(x) = x + h)$.

Now it is not difficult to show, that

$$\Phi \circ \Phi_{h_0} \circ \Phi^{-1} \in A(E) \setminus Q_\mu.$$

This ends the proof of Theorem 3.33. $\square$

Chapter 4

Borel Product-Measures in $\mathbf{R}^I$

We start our discussion with standard notions and definitions from probability theory.

Let I be an arbitrary nonempty set of parameters. Denote by $(\mathbf{R}^I, \tau)$ the vector space of all real-valued functions on I equipped with Tykhonoff topology τ. Assume that $B(\mathbf{R}^I)$ is the σ-algebra of all Borel subsets of the space $\mathbf{R}^I$, generated by the Tykhonoff topology τ;

Let $(Pr_i)_{i\in I}$ be the family of projections defined by

$$(\forall i)(\forall (x_j)_{j\in I})(i \in I \ \& \ (x_j)_{j\in I} \in \mathbf{R}^I \to Pr_i((x_j)_{j\in I}) = x_i).$$

A minimal σ-algebra of subsets of $\mathbf{R}^I$ generated by the class of subsets

$$(Pr_i^{-1}(X))_{i\in I, X\in B(\mathbf{R})}$$

is denoted by $Ba(\mathbf{R}^I)$ and is called a Baire σ-algebra of subsets of $\mathbf{R}^I$.

Remark 4.1 Note that $Ba(\mathbf{R}^I) = B(\mathbf{R}^I)$ for $\mathrm{card}(I) \leq \aleph_0$, where $\aleph_0$ is the cardinality of the set of all natural numbers. If $\mathrm{card}(I) > \aleph_0$, then

$$Ba(\mathbf{R}^I) \subset B(\mathbf{R}^I) \ \& \ B(\mathbf{R}^I) \setminus Ba(\mathbf{R}^I) \neq \emptyset.$$

As usual, a measure defined on $B(\mathbf{R}^I)$ is called a Borel measure. Analogously, a measure defined on $Ba(\mathbf{R}^I)$ is called a Baire measure.

Denote also

$$\mathbf{R}^{(I)} = \{(x_t)_{t\in I} : (x_t)_{t\in I} \in \mathbf{R}^I \& \mathrm{card}\{i | x_i \neq 0\} < \aleph_0\}.$$

Definition 4.1 A Borel probability measure μ defined on $(\mathbf{R}^I, \tau)$ is called Radon if

$$(\forall B)(B \in B(X) \to \mu(B) = \sup\{\mu(K) \ : \ K \subset B \ \& \ K \text{ is compact in } \mathbf{R}^I\}).$$

Definition 4.2 A family $(U_i)_{i\in I}$ of open subsets in $(\mathbf{R}^I, \tau)$ is called a generalized sequence if

$$(\forall i_1)(\forall i_2)(i_1 \in I \ \& \ i_2 \in I \to (\exists i_3)(i_3 \in I \to (U_{i_1} \subset U_{i_3} \ \& \ U_{i_2} \subset U_{i_3}))).$$

Definition 4.3 A Borel probability measure μ defined on $(\mathbf{R}^I,\tau)$ is called τ-smooth if, for an arbitrary generalized sequence $(U_i)_{i\in I}$, the condition

$$\mu(\bigcup_{i\in I} U_i) = \sup_{i\in I} \mu(U_i)$$

is valid.

Definition 4.4 A Baire probability measure μ on $\mathbf{R}^I$ is called τ-smooth if, for an arbitrary generalized sequence $(U_i)_{i\in I}$ of open Baire subsets in $\mathbf{R}^I$, for which $\bigcup_{i\in I} U_i$ is also a Baire subset, the condition

$$\mu(\bigcup_{i\in I} U_i) = \sup_{i\in I} \mu(U_i)$$

is valid (see e.g. [173]).

Definition 4.5 Let μ_1 be a Baire measure defined on $\mathbf{R}^I$. A Borel measure μ_2 defined on $\mathbf{R}^I$ is called a Borel extension of μ_1 if

$$(\forall X)(X \in Ba(\mathbf{R}^I) \rightarrow \mu_2(X) = \mu_1(X)).$$

Example 4.1 Let I be an arbitrary nonempty parametric set, and p_i be a Borel probability measure on $\mathbf{R}$ for all $i \in I$. If $\mathrm{card}(I) > \aleph_0$, then the probability product-measure $\prod_{i\in I} p_i$ is defined on the σ-algebra

$$\prod_{i\in I} B(R_i) = Ba(\mathbf{R}^I).$$

Accordingly, this measure is an example of a Baire probability measure which is not defined on $B(\mathbf{R}^I)$.

The following definition from general topology is important for our investigation.

Definition 4.6 Assume that for arbitrary $i \in I$, f_i is a mapping of the topological space $({X_i}^{(1)}, {\tau_i}^{(1)})$ into the topological space $({X_i}^{(2)}, {\tau_i}^{(2)})$. Then the mapping $\prod_{i\in I} f_i$ defined by

$$(\forall (x_i)_{i\in I})((x_i)_{i\in I} \in \prod_{i\in I} {X_i}^{(1)} \rightarrow \ (\prod_{i\in I} f_i\ )\ ((x_i)_{i\in I}) = (\ f_i(x_i))_{i\in I})$$

is called a direct product of the family $(\ f_i)_{i\in I}$ (see e.g. [34]).

In the sequel, we will need some auxiliary results.

Lemma 4.1 *If a Baire measure P on $\mathbf{R}^I$ is τ-smooth, then there exists only one τ-smooth Borel extension of P on $\mathbf{R}^I$.*

The proof of Lemma 4.1 can be found in [173].

Lemma 4.2 *Let f_i be a mapping of the topological space $(X_i^{(1)}, \tau_i^{(1)})$ into the topological space $(X_i^{(2)}, \tau_i^{(2)})$ for $i \in I$. Then the direct product $\prod_{i\in I} f_i$ of the family $(\ f_i)_{i\in I}$ is continuous if and only if*

$$(\forall i)(i \in I \rightarrow f_i \textit{ is continuous}).$$

The proof of Lemma 4.2 is easy and is given e.g. in [34].

Remark 4.2 It is easy to verify that if $\prod_{i\in I} f_i$ is a continuous mapping, then $\prod_{i\in I} f_i$ is also a Borel measurable mapping.

Remark 4.3 If a mapping f_i is a $(B({X_i}^{(2)}), B({X_i}^{(1)}))$-measurable for an arbitrary $i \in I$, then $\prod_{i\in I} f_i$ is $(Ba(\prod_{i\in I}{X_i}^{(2)}), Ba(\prod_{i\in I}{X_i}^{(1)}))$-measurable mapping, where $B({X_i}^{(k)})$ denotes the σ-algebra of subsets of ${X_i}^{(k)}$ generated by the topology ${\tau_i}^{(k)}(k=1,2)$; by $Ba(\prod_{i\in I}{X_i}^{(k)})$ is denoted the Baire σ-algebra of subsets of the product topological space

$$(\prod_{i\in I}{X_i}^{(k)}, \prod_{i\in I}{\tau_i}^{(k)})(k=1,2).$$

Lemma 4.3 *Let p_i be a probability Borel measure defined on $\mathbf{R}$ for an arbitrary $i \in I$. Then there exists a family $(f_i)_{i\in I}$ of Borel measurable real functions defined on $]0;1[$ such that*

$$(\forall i)(\forall y)(i \in I \ \&\ y \in \mathbf{R} \to l_1(\{x|x \in]0;1[\ \&\ f_i(x) \le y\}) = p_i((-\infty;y])\ ),$$

where l_1 denotes the Lebesgue measure on $]0;1[$.

This lemma is well known and its proof can be found e.g. in [152](cf.Example 3.25).

Lemma 4.4 *Let (E_1,τ_1) and (E_2,τ_2) be two topological spaces. Denote by $B(E_1)$ and $B(E_2)$ (correspondingly, $B(E_1 \times E_2)$) the class of all Borel subsets generated by the topologies τ_1 and τ_2 (correspondingly, $\tau_1 \times \tau_2$). If at least one of these topological spaces has a countable base, then the equality*

$$B(E_1) \times B(E_2) = B(E_1 \times E_2)$$

holds.

Proof. Let E_1 has a countable base of open sets. Denote by $(B_n)_{n\in N}$ some base of this space. Lemma 4.4 will be proved if we show that an arbitrary open set in $E_1 \times E_2$ can be expressed by the union

$$\cup_{n\in N} A_n,$$

where, for an arbitrary $n \in N$, A_n is an elementary open set in $E_1 \times E_2$. The latter means that, for A_n, we have the representation

$$A_n = {A_n}^{(1)} \times {A_n}^{(2)},$$

where ${A_n}^{(k)}$ is an open set in $E_k(k=1,2)$.

Obviously, we can write

$$G = \cup_{t\in I} U_t$$

for an arbitrary open set G in $E_1 \times E_2$, where

a) I is some set of parameters;

b) $(\forall t)(\forall k)(t \in I\ \&\ (k=1,2) \to ({U_t}^{(k)}$ is an open set in $E^{(k)}))$.

For an arbitrary parameter $t \in I$ denote by θ_t the set of all natural numbers for which we have

$$\tilde{U}_n(t) = \cup_{n\in\theta_t} B_n \times {U_t}^{(2)}.$$

Let us put

$$\tilde{U}_t = \begin{cases} U_t^{(2)}, & \text{if } B_n \times U_t^{(2)} \subset G; \\ \emptyset, & \text{if } B_n \times U_t^{(2)} \not\subset G. \end{cases}$$

On the one hand, if $x \in G = \cup_{t \in I} U_t$, then there exist $t_0 \in I$ and $n_0 \in \theta_{t_0}$ such that

$$x \in B_{n_0} \times {U_{t_0}}^{(2)}.$$

This means that

$$B_{n_0} \times U_{t_0}^{(2)} = B_{n_0} \times \tilde{U}_{n_0}(t_0) \subset \cup_{n \in N} (B_n \times \cup_{t \in I} \tilde{U}_n(t)).$$

On the other hand, if

$$x \in \cup_{n \in N} (B_n \times \cup_{t \in I} \tilde{U}(t)),$$

then there exist $n_0 \in N$ and $t_0 \in I$ such that

$$x \in B_{n_0} \times \tilde{U}_{n_0}(t_0).$$

By the definition of the set $\tilde{U}_{n_0}(t)$ we have

$$B_{n_0} \times {U_{t_0}}^{(2)} \subset G.$$

This completes the proof of Lemma 4.4. □

Denote by S_i the unit circle S in the Euclidean plane $\mathbf{R}^2$. We can identify the circle S_i with a compact group of all rotations of $\mathbf{R}^2$ about its origin.

Lemma 4.5 *Assume that λ_I is a probability Haar measure defined on the group $\prod_{i \in I} S_i$. Then, for $card(I) > \aleph_0$, the set X defined by*

$$X = \prod_{i \in I} (S_i \setminus \{(0;1)_i\})$$

is a λ_I-massive nonmeasurable subset of the group $\prod_{i \in I} S_i$.

Proof. Let us show that the inner λ_I-measure of the set X is equal to zero. If we assume the contrary, then by using the inner regularity of the Haar measure (see [51]) we obtain the existence of a compact subset $F \subset X$ such that $\lambda_I(F) > 0$. It is clear that $\lambda_i(Pr_i(F)) < 1$, where λ_i is the probability Haar measure on S_i and Pr_i is the projection from $\prod_{j \in I} S_j$ onto S_i.

We have

$$\lambda_I(\prod_{i \in I} Pr_i(F)) = 0$$

because $Card(I) > \aleph_0$. This contradicts the conditions

$$F \subset \prod_{i \in I} Pr_i(F),\ \lambda_I(F) > 0.$$

Let us show that

$$\lambda_I^*(\prod_{i \in I} (S_i \setminus \{(0.1)_i\})) = 1.$$

Assume the contrary and let

$$\lambda_I^*(\prod_{i\in I}(S_i\setminus\{(0.1)_i\}))<1.$$

Then there exists $B\in B(\prod_{i\in I}S_i)$ such that

$$\prod_{i\in I}(S_i\setminus\{0,1\})\subset B$$

and $0\leq\lambda(B)<1$. On the other hand we have

$$\lambda(B)=\inf_{B\subset U,U\in Ba(\prod_{i\in I}S_i)}\lambda_*(U)=\inf_{B\subset U,U\in Ba(\prod_{i\in I}S_i)}\lambda(U).$$

Let U_k be such element of $Ba(\prod_{i\in I}S_i)$ that
1) $B\subset U_k$;
2) $\lambda(B)<\lambda(U_k)<\lambda(B)+\frac{1}{k}$.
It is clear, that $\cap_{k\in N}U_k\in Ba(\prod_{i\in I}S_i)$ and $B\subset\cap_{k\in N}U_k$, $\lambda(\cap_{k\in N}U_k)=\lambda(B)<1$.
Clearly, $\cap_{k\in N}U_k=U_J\times\prod_{i\in I\setminus J}S_i$ for some $J\subset I$ with $\mathrm{card}(J)\leq\aleph_0$. We have

$$\prod_{i\in I}(S_i\setminus\{(0.1)_i\})\subset B\subset U_J\times\prod_{i\in I\setminus J}S_i.$$

Obviously

$$Pr_J(\prod_{i\in I}(S_i\setminus\{(0.1)_i\}))\subset U_J$$

and $\lambda_J(U_J)<1$, because $\lambda(\cap_{k\in N}U_k)=\lambda(U_J\times\prod_{i\in I\setminus J}S_i)=\lambda_J(U_J)$. But this relation is not possible because

$$Pr_J(\prod_{i\in I}(S_i\setminus\{(0.1)_i\}))=\prod_{i\in J}(S_i\setminus\{(0.1)_i\})\subset U_J$$

and

$$\lambda_J(\prod_{i\in I}(S_i\setminus\{(0.1)_i\}))=1.$$

This finishes the proof of Lemma 4.5. $\square$

Remark 4.4 The results of lemmas 4.4 and 4.5 were obtained in [88].

Define the measure $\lambda_{(I)}$ by

$$(\forall B)(B\in\prod_{i\in I}S_i\rightarrow\lambda_{(I)}(\prod_{i\in I}(S_i\setminus\{(0;1)_i\}\cap B))=\lambda_I(B)).$$

Denote by $\tilde{\lambda}_{(I)}$ the completion of the measure $\lambda_{(I)}$ and define μ_I by

$$(\forall X)(X\in\text{ dom }(\tilde{\lambda}_{(I)})\rightarrow\mu_I((\prod_{i\in I}g_i)^{-1}(X))=\tilde{\lambda}_I(X)),$$

where

$$(\forall x)(x\in]0;1[\ \rightarrow g_i(x)=(-\sin(2\pi x),\cos(2\pi x))\ ).$$

Denote by $\tilde{B}(]0;1[^I)$ the σ -algebra of μ_I-measurable subsets of $]0;1[^I$.

The following result is of some interest.

Theorem 4.1 *Let $(p_i)_{i\in I}$ be an arbitrary family of Borel probability measures defined on $\mathbf{R}$. Let $(f_i)_{i\in I}$ be the family of Borel measurable functions constructed in Lemma 4.3. If the mapping $\prod_{i\in I} f_i$ is a $(B(\mathbf{R}^I),\tilde{B}(]0;1[^I))$-measurable mapping, then in $\mathbf{R}^I$ there exists a Borel extension $\mathcal{P}_I$ of the Baire probability product-measure $\prod_{i\in I} p_i$.*

Proof. Define the functional $\mathcal{P}_I$ by

$$(\forall X)(X\in B(\mathbf{R}^I)\rightarrow \mathcal{P}_I(X)=\mu_I((\prod_{i\in I} f_i)^{-1}(X))).$$

Show that the measure $\mathcal{P}_I$ is an extension of the product-measure $\prod_{i\in I} p_i$. Indeed, for every finite parameter set $I_0\subset I$ and for a Baire set

$$\prod_{i\in I_0}(-\infty;x_i]\ (i\in I_0, x_i\in\mathbf{R})$$

we have

$$\mathcal{P}_I(R^{I\setminus I_0}\times\prod_{i\in I_0}(-\infty;x_i])=\mu_I((\prod_{i\in I} f_i)^{-1}(\mathbf{R}^{I\setminus I_0}\times\prod_{i\in I_0}(-\infty;x_i])$$

$$=\mu_I(]0;1[^{I\setminus I_0}\times\prod_{i\in I_0}\{y: y\in\,]0;1[\ \&\ f_i(y)\le x_i\}).$$

In view of lemmas 4.3 and 4.4, using the property of a Haar measure, we have

$$\mu_I(]0;1[^{I\setminus I_0}\times\prod_{i\in I_0}\{y: y\in]0;1[\ \&\ f_i(y)\le x_i\})=$$

$$=\mu_{I\setminus I_0}(]0;1[^{I\setminus I_0})\times\mu_{I_0}(\prod_{i\in I_0}\{y: y\in]0;1[\ \&\ f_i(y)\le x_i\})=$$

$$=(\prod_{i\in I_0} l_1)(\{y: y\in]0;1]\ \&\ f_i(y)\le x_i\})=\prod_{i\in I_0}F_i(x)=(\prod_{i\in I} p_i)\ (R^{I\setminus I_0}\times\prod_{i\in I_0}(-\infty;x_i]).$$

This completes the proof of Theorem 4.1. □

Remark 4.5 From the results of Theorem 4.1 and lemmas 4.1–4.4 we conclude that for an arbitrary family $(p_i)_{i\in I}$ of quasiinvariant Borel probability measures defined on $\mathbf{R}$, there exists only one Borel extension $\mathcal{P}_I$ of the probability Baire product-measure $\prod_{i\in I} p_i$.

Remark 4.6 The mapping $\prod_{i\in I} f_i$ is always $(Ba(\mathbf{R}^I),Ba(]0;1[^I))$-measurable and so we have the following representation of the Baire product measure $\prod_{i\in I} p_i$ constructed by Anderson:

$$(\forall B)(B\in Ba(\mathbf{R}^I)\rightarrow(\prod_{i\in I} p_i)(B)=\mu_I((\prod_{i\in I} f_i)^{-1}(B))).$$

An example of a $R^{(I)}$-quasiinvariant Borel extension of a concrete Baire probability product-measure on $\mathbf{R}^I$ has been constructed in [88]. Hence it is of interest to point out,

within the class of all Borel extensions of the probability product-measures defined on $\mathbf{R}^I$, a subclass of measures whose every element possesses the property of $R^{(I)}$-quasiinvariance.

We will need the following lemma.

Lemma 4.6 *Let I be some nonempty set of parameters and, for an arbitrary parameter $i \in I$, let S_i be the unit circle in the Euclidean plane $\mathbf{R}^2$, and let $(p_i)_{i\in I}$ be a family of quasiinvariant Borel probability measures on $\mathbf{R}_i = R$ with distribution functions $(F_i)_{i\in I}$.*

For $i \in I$ denote by f_i the mapping of $\mathbf{R}_i$ into $S_i \setminus \{(0;1)_i\}$ defined by

$$(\forall x)(x \in \mathbf{R}_i \to f_i(x) = (-sin(2\pi F_i(x)), \cos(2\pi F_i(x))).$$

If h is an arbitrary translation of the real axis $\mathbf{R}_i$, then it is clear that the mapping $f_i \circ h \circ f_i^{-1}$ is a continuous automorphism of the space $S_i \setminus \{(0;1)_i\}$. This automorphism is uniquely extended to a continuous automorphism of the unit circle. Denote by G_i the group of all such automorphisms of the unit circle. Then the probability Haar measure λ_I defined on $\prod_{i\in I} S_i$ is quasiinvariant with respect to the direct sum $\sum_{i\in I} G_i$ of the family of groups $(G_i)_{i\in I}$.

Proof. Denote by λ_i the probability Haar measure defined on the unit circle S_i $(i \in I)$. We will prove the lemma in two steps.

1) The measure λ_i is G_i-quasiinvariant. Note that, for an arbitrary neighbourhood U of the point $(0;1)_i$ and for an arbitrary element $g \in G_i$, there exists a positive real number $K(U,g,i)$ such that

$$(\forall x_1)(\forall x_2)((x_1 \in S_i \setminus U) \ \& \ (x_2 \in S_i \setminus U) \to$$
$$\lambda_i(\overline{(g(x_1),g(x_2))} \leq K(U,g,i) \times \lambda_i(\overline{(x_1,x_2)})),$$

where $\overline{(x_1,x_2)}$ and $\overline{(g(x_1),g(x_2))}$ denote the open arcs of the unit circle S_i which do not contain the point $(0;1)_i$.

Indeed, without loss of generality we can consider a neighbourhood U of the point $(0;1)_i$ having the form

$$(\exists \alpha_0)(0 < \alpha_0 < \pi \to U = \{(x;y) | (x;y) = (cos\phi, sin\phi), \frac{\pi}{2} - \alpha_0 < \phi < \frac{\pi}{2} + \alpha_0\}.$$

Let $x_1 \in S_i \setminus U$ and $x_2 \in S_i \setminus U$. Then for $g \in G_i$ there exists a translation h of $\mathbf{R}_i$ such that

$$g = f_i \circ h \circ f_i^{-1}.$$

We have

$$\lambda(\overline{g(x_1,x_2)}) = \int_{t_1+h}^{t_2+h} F_i'(t)d(t) = (t_2 - t_1) \times F_i'(\xi_1) =$$

$$= (t_2 - t_1) \times \frac{F_i'(\xi_1)}{F_i'(\xi_2)} \times F_i'(\xi_2) \leq (t_2 - t_1) \times \frac{\sup_{t\in[t_1+h;t_2+h]} F_i'(t)}{\inf_{t\in[t_1;t_2]} F_i'(t)} \times F_i'(\xi_2) \leq$$

$$\leq K(U,g,i)\times\lambda_i(\overline{(x_1,x_2)}),$$

where

$$x_1=f_i(t_1),\ x_2=f_i(t_2)\ (t_1<t_2)\ ;\ t_1+h<\xi_1<t_2+h,\ t_1<\xi_2<t_2,$$

$$\int_{t_1}^{t_2} F_i'(x)dx=(t_2-t_1)\times F'(\xi_2),$$

$$K(U,g,i)=\frac{\sup_{t\in[f_i^{-1}(t_1^0)-|h|,f_i^{-1}(t_2^0)+|h|]} F_i'(t)}{\inf_{t\in f_i^{-1}(S_i\setminus U)} F_i'(t)},$$

$$f_i(t_1^0)=(\cos(\frac{\pi}{2}-\alpha_0),\ sin(\frac{\pi}{2}-\alpha_0)),\ f_i(t_2^0)=(\cos(\frac{\pi}{2}+\alpha_0),\sin(\frac{\pi}{2}+\alpha_0)).$$

Let X be an arbitrary λ_i-measure zero subset of the space S_i. Let $(U_n)_{n\in N}$ be a fundamental system of neighbourhoods of the point $(0;1)_i$.

For an arbitrary element $g\in G_i$, consider the set $g(X)$. We must prove that the set $g(X)$ is λ_i-measure zero.

Let us put

$$X_n=(S_i\setminus U_n)\cap X\ (n\in N).$$

The relation

$$g(X)=\begin{cases}\cup_{n\in N}\ g(X_n)\cup\{(0;1)_i\}, & \text{if } (0;1)_i\in X;\\ \cup_{n\in N}\ g(X_n), & \text{if } (0;1)_i\notin X\end{cases}$$

holds.

It is sufficient to show that

$$(\forall n)(n\in N\rightarrow\lambda_i(g(X_n))=0).$$

Let ε be an arbitrary positive number. Then by the property of a Haar measure (see e.g.[54]), there exists, in S_i, an open set G such that

$$X_n\subset G\subset S_i\setminus U_n,\ \lambda_i(G\setminus X_n)<\frac{\varepsilon}{K(U,g,i)}.$$

By using the structure of open sets in the space S_i, we deduce the existence of disjoint arcs $\{\overline{(a_k,b_k)}\}_{k\in N}$ such that:

$$A)\ \overline{(a_k,b_k)}\subset S_i\setminus U_n\ (i\in I,k\in N);$$

$$B)\ G=\bigcup_{k\in N}\overline{(a_k,b_k)}.$$

Clearly, we have

$$g(X_n)\subset g(G)\ \&\ \lambda_i(g(G))=\lambda_i(g(\cup_{k\in N}\overline{(a_k,b_k)}))=\sum_{k\in N}\lambda_i(g\overline{(a_k,b_k)})\leq$$

$$\leq \sum_{k\in N} K(U_n,g,i)\times\lambda_i(\overline{(a_k,b_k)}) = K(U_n,g,i)\times\sum_{k\in N}\lambda_i(\overline{(a_k,b_k)}) =$$

$$K(U_n,g,i)\times\lambda_i(G)\leq K(U_n,g,i)\times\frac{\varepsilon}{K(U_n,g,i)}=\varepsilon.$$

This means that λ_i is G_i-quasiinvariant.

2) The measure λ_I is $\sum_{i\in I} G_i$-quasiinvariant.

Note that we have to prove

$$(\forall B)(\forall g)(B\in B(\prod_{i\in I} S_i)\ \&\ g\in\sum_{i\in I} G_i \rightarrow (\lambda_I(B)=0\Leftrightarrow\lambda_I(g(B))=0)\ ).$$

For

$$g=(g_i)_{i\in I}\in\sum_{i\in I} G_i$$

there exists a subset $I_0\subset I$ such that

$$A)\ \mathrm{Card}(I_0)<\aleph_0.$$

$$B)\ (\forall i)(i\in(I\setminus I_0)\rightarrow g_i=e_i),$$

where e_i is the unit element of the group G_i $(i\in I)$.

Using the result of Lemma 4.4, we can easily verify the validity of the equality

$$\lambda_I=\prod_{i\in I_0}\lambda_i\ \times\ \lambda_{I\setminus I_0},$$

where $\lambda_{I\setminus I_0}$ is the probability Haar measure defined on the topological group $\prod_{i\in I\setminus I_0} S_i$. If we consider the measure $\lambda_{I\setminus I_0}$ as a $(e_i)_{i\in I\setminus I_0}$-quasiinvariant measure, then in virtue of the well-known result of Kakutani about the product of quasiinvariant measures we easily verify the validity of Lemma 4.6. $\square$

A general result is contained in the following theorem.

Theorem 4.2 *Let I be an arbitrary nonempty set of parameters, $(p_i)_{i\in I}$ be a family of quasiinvariant Borel probability measures defined on the real axis $\mathbf{R}$. Then in the space $\mathbf{R}^I$ there exists only one Borel extension of the product-measure $\prod_{i\in I} p_i$ which is quasiinvariant under the group $\mathbf{R}^{(I)}$.*

Proof. Denote by μ_I the functional defined by

$$(\forall B)(B\in B(\prod_{i\in I} S_i)\rightarrow\mu_I(B\cap\prod_{i\in I}(S_i\setminus(0;1)_i))=\lambda_I(B)).$$

First, let us show the correctness of this definition.

Assume that for two different Borel subsets $B_1\in B(\prod_{i\in I} S_i)$ and $B_2\in B(\prod_{i\in I} S_i)$ we have

$$B_1\cap(\prod_{i\in I}(S_i\setminus\{(0;1)_i\}))=B_2\cap(\prod_{i\in I}(S_i\setminus\{(0;1)_i\})).$$

Suppose also that $\lambda_I(B_1 \triangle B_2) > 0$. Without loss of generality we may assume that

$$\lambda_I(B_1 \setminus B_2) > 0.$$

By the property of the massive set $\prod_{i\in I}(S_i \setminus \{(0;1)_i\})$, we have

$$(B_1 \setminus B_2) \cap (\prod_{i\in I}(S_i \setminus \{(0;1)_i\})) \neq \emptyset,$$

which is a contradiction, and therefore the correctness of the definition of the measure μ_I is proved.

It is clear that the measure μ_I is $\sum_{i\in I} G_i$-quasiinvariant.

Denote by $\mathcal{P}_I$ the functional defined by

$$(\forall B)(B \in B(\mathbf{R}^I) \to \mathcal{P}_I(B) = \mu_I((\prod_{i\in I} f_i)(B)),$$

where $(f_i)_{i\in I}$ is the family of mappings constructed in Lemma 4.6.

If B is an arbitrary $\mathcal{P}_I$ -measure zero subset of the space $\mathbf{R}^I$, then we have

$$(\forall h)(h \in \sum_{i\in I} R_i \to \mathcal{P}_I(h(B)) = \mu_I((\prod_{i\in I} f_i)(h(B))) =$$

$$= \mu_I((\prod_{i\in I} f_i) \circ h^{-1} \circ (\prod_{i\in I} f_i)^{-1} \circ (\prod_{i\in I} f_i) \circ h(B)) = \mu_I((\prod_{i\in I} f_i)(B)) = \mathcal{P}_I(B)).$$

Note that for every finite parameter set I_0 and for every Baire set

$$\prod_{i\in I_0}(-\infty; x_i] \ (x_i \in \mathbf{R}_i, i \in I_0)$$

we have

$$\mathcal{P}_I(\mathbf{R}^{I\setminus I_0} \times \prod_{i\in I_0}(-\infty; x_i]) = \mu_I((\prod_{i\in I} f_i)(\mathbf{R}^{I\setminus I_0} \times \prod_{i\in I_0}(-\infty; x_i]) =$$

$$= \mu_I(\prod_{j\in I\setminus I_0}(S_j \setminus \{(0;1)_j\}) \times (\prod_{i\in I_0} f_i)(\prod_{i\in I_0}(-\infty; x_i])) =$$

$$= \mu_I(\prod_{j\in I\setminus I_0}(S_j \setminus \{(0;1)_i\}) \times \prod_{i\in I_0} f_i((-\infty; x_i])) = \lambda_I(\prod_{j\in I\setminus I_0} S_j \times \prod_{i\in I_0} f_i((-\infty; x_i])) =$$

$$= \lambda_{I\setminus I_0}(\prod_{i\in I\setminus I_0} S_i) \times \lambda_{I_0}(\prod_{i\in I_0} f_i((-\infty; x_i])) = \lambda_{I_0}(\prod_{i\in I_0} f_i((-\infty; x_i])) =$$

$$= \prod_{i\in I_0} \int_{-\infty}^{x_i} F_i'(x)dx = \prod_{i\in I_0} F_i(x_i) = \prod_{i\in I} p_i(\mathbf{R}^{I\setminus I_0} \times \prod_{i\in I_0}(-\infty; x_i]).$$

This means that the measure $\mathcal{P}_I$ is a Borel extension of the Baire product measure $\prod_{i\in I} p_i$.

By the property of τ_0-smoothness of the Haar measure λ_I (see e.g. [54]), where τ_0 denotes the topology on $\prod_{i\in I} S_i$, we conclude that the measure μ_I is τ_1-smooth, where τ_1

denotes the induced (by τ_0)topology on the space $\prod_{i\in I}(S_i \setminus \{(0;1)_i\})$. Analogously, by the property of τ_1-smoothness of the probability measure μ_I and by the equality

$$\mathcal{P}_I = (\prod_{i\in I} f_i)^{-1} \circ \mu_I,$$

we conclude that the measure $\mathcal{P}_I$ is τ-smooth, where τ denotes the Tykhonoff's topology in the space $\mathbf{R}^I$.

Finally, by the property of τ-smoothness of $\mathcal{P}_I$, the product measure $\prod_{i\in I} p_i$ is also τ-smooth and, using the result of Theorem 4.1, we conclude that the Borel extension $\mathcal{P}_I$ of the product measure $\prod_{i\in I} p_i$ is unique.

This ends the proof of Theorem 4.2. $\square$

Remark 4.7 When we extend a quasiinvariant measure from one class of subsets to the another no always is possible to preserve a property of quasiinvariance. In this context see Remark 1.1.

Let us consider some corollaries of Theorem 4.2.

Corollary 4.1 The main result of [88] can be obtained if we assume

$$(\forall i)(\forall x)(i \in I \;\&\; x \in \mathbf{R} \rightarrow F_i(x) = \frac{1}{2} + \frac{1}{\pi} \cdot arctg(\frac{x}{2})).$$

Corollary 4.2 The product of an arbitrary family $(p_i)_{i\in I}$ of nontrivial Gaussian Borel probability measures defined on $\mathbf{R}^I$ has only one Borel extension which is quasiinvariant with respect to the vector subspace $\mathbf{R}^{(I)}$. (cf. the proof of the general result obtained in [171]).

Corollary 4.3 In the case of the space $\mathbf{R}^I$, for $Card(I) > \aleph_0$ Theorem 4.2 is a generalization of the Anderson well known theorem which gives only the construction of quasiinvariant Baire product-measures

Note that, for $Card(I) > \aleph_0$, in the vector space $\mathbf{R}^I$ there exists no Radon probability measure which would be quasiinvariant with respect to the vector subspace $\mathbf{R}^{(I)}$ (see Chapter 6, also [130]).

For $Card(I) > 2^{\aleph_0}$ in the vector space $\mathbf{R}^I$ there exists no Radon probability measure which would be quasiinvariant with respect to an everywhere dense subspace of $\mathbf{R}^I$ (see [130], also Chapter 6).

If $Card(I) = 2^{\aleph_0}$, then in the vector space $\mathbf{R}^I$ there exists a nontrivial σ-finite Radon measure which is invariant with respect to some everywhere dense subspace of $\mathbf{R}^I$ (see [130]).

Here we present the following two unsolved problems:

Problem 4.1 Does there exist a nontrivial σ-finite Borel measure on the space $\mathbf{R}^I$ for $Card(I) \geq \aleph_1$ which would be invariant with respect to the vector subspace $\mathbf{R}^{(I)}$ of the space $\mathbf{R}^I$?

Problem 4.2 Does there exist a nontrivial σ-finite Borel measure on the space $\mathbf{R}^I$ for $Card(I) > 2^{\aleph_0}$ which would be invariant with respect to some everywhere dense vector subspace of $\mathbf{R}^I$?

In connection with these problems, see [91].

Let I be an arbitrary infinite parameter set, and μ be a canonical probability Gaussian Borel measure defined on $\mathbf{R}^I$. The following natural problem arises: give a characterization of the vector space of all admissible translations of the Borel probability product-measures defined on $\mathbf{R}^I$. The solution of this problem is given below.

Let $X = \prod_{k \in N} X_k$ be the product of countable measurable spaces, and let μ_k and ν_k $(k \in N)$ be probability measures such that:

1) μ_k is absolutely continuous with respect to ν_k,

2) $\frac{d\mu_k(x)}{d\nu_k(x)} = \rho_k(x)$.

Let us consider the product-measures $\mu = \prod_{k \in N} \mu_k$ and $\nu = \prod_{k \in N} \nu_k$.

The following important statement due to Kakutani is valid.

Theorem 4.3 (Kakutani) *The measures μ and ν are equivalent if and only if an infinite product $\prod_{k \in N} \alpha_k$ is divergent to zero, where $\alpha_k = \int_{X_k} \sqrt{\rho_k(x_k)} d\nu_k(x_k)$. In this case $r_n(x) = \prod_{k=1}^{n} \rho_k(x)$ is convergent (in the mean) to the function $r(x) = \prod_{k=1}^{\infty} \rho_k(x)$ which is the density of the measure μ with respect to ν, i.e.,*

$$r(x) = \frac{d\mu(x)}{d\nu(x)}.$$

Proof. Let us show that if $\prod_{k \in N} \alpha_k$ is convergent to zero, then the measures μ and ν are orthogonal. So far as

$$\alpha_k^2 = |\int_{X_k} \sqrt{\rho_k(x_k)} d\nu_k(x_k)|^2 \leq \int_{X_k} \rho_k(x_k) d\nu(x_k) = 1,$$

the product $\prod_{k \in N} \alpha_k$ cannot be convergent to infinity. If this product is convergent to zero, then there exists a sequence $\beta_s = \prod_{k=n_s}^{m_s} \alpha_k$ such that the series $\sum_{s=1}^{\infty} \beta_s$ is convergent.

Let us consider the sequence of Baire sets

$$A_s = \{x : \prod_{k=n_s}^{m_s} \rho_k(x_k) \geq 1\}.$$

Note that

$$\nu(A_s) = \int_{A_s} d\nu(x) \leq \int_{A_s} \sqrt{\prod_{k=n_s}^{m_s} \rho_k(x_k)} d\nu(x) =$$

$$= \prod_{k=n_s}^{m_s} \int_{X_k} \sqrt{\rho_k(x_k)} d\nu_k(x_k) = \beta_s.$$

From the convergence of the series $\sum_{s=1}^{\infty} \beta_s$ we have

$$\nu(A) = 0,$$

where

$$A = \overline{\lim}_{s\to\infty} A_s.$$

On the other hand, if $B_s = X \setminus A_s$, then

$$\mu(B_s) = \int_{B_s} d\mu(x) \leq \int_{B_s} \{\prod_{k=n_s}^{m_s} \rho_k(x_k)\}^{-\frac{1}{2}} d\mu(x) =$$

$$= \int_{B_s} \sqrt{\prod_{k=n_s}^{m_s} \rho(x_k)} d\nu(x) \leq \prod_{k=n_s}^{m_s} \int_{X_k} \sqrt{\rho_k(x_k)} d\nu_k(x_k) = \beta_s.$$

Hence $\mu(\overline{\lim}_{s\to\infty} B_s) = 0$. The latter relation implies $\mu(A) \geq \mu(\underline{\lim} A_s) = 1$, i.e., $\mu(A) = 1$ and we conclude that $\mu \perp \nu$.

Now, assume that the product $\prod_{k\in N} \alpha_k$ is divergent to zero. Let us consider the sequence of functions $(\Phi_n)_{n\in N}$, where

$$\Phi_n(x) = \sqrt{\prod_{k=1}^{n} \rho_k(x_k)}\ (n \in N).$$

From the validity of the relation

$$\int_X |\Phi_{n+p}(x) - \Phi_n(x)|^2 d\nu(x) = \int_X \prod_{k=1}^{n} \rho_k(x_k) |\sqrt{\prod_{k=n+1}^{n+p} \rho_k(x_k)} - 1|^2 d\nu(x) =$$

$$= \int_{X_{n+1}} \cdots \int_{X_{n+p}} |\sqrt{\prod_{k=n+1}^{n+p} \rho_k(x_k)} - 1|^2 \prod_{k=n+1}^{n+p} d\nu_k(x_k) = 2(1 - \prod_{k=n+1}^{n+p} \alpha_k),$$

we easily deduce that $\Phi_n(x)$ is a fundamental sequence in $L_2(X,\nu)$.

So far as

$$\int_X |r_{n+p}(x) - r_n(x)| d\nu(x) \leq \{\int_X |\Phi_{n+p}(x) - \Phi_n(x)|^2 d\nu(x)\}^{\frac{1}{2}} \times$$

$$\times \{\int_X |\Phi_{n+p}(x) + \Phi_n(x)|^2 d\nu(x)\}^{\frac{1}{2}} \leq$$

$$\leq 2\{\int_X |\Phi_{n+p}(x) - \Phi_n(x)|^2 d\nu(x)\}^{\frac{1}{2}},$$

the sequence $(r_n)_{n\in N} = (\Phi_n^2)_{n\in N}$ is convergent in the mean.

Let $r(x) = \lim_{n\to\infty} r_n(x)$. For an arbitrary bounded Baire function f, the equalities

$$\int_X f(x)d\mu(x) = \lim_{n\to\infty}\int_X f(x_1,\cdots,x_n)\prod_{k=1}^{n} d\mu_k(x_k) =$$

$$= \lim_{n\to\infty}\int_X f(x_1,\cdots,x_k)r_n(x)\prod_{k=1}^{n} d\nu_k(x) = \lim_{n\to\infty}\int_X f(x)r_n(x)d\nu(x) =$$

$$= \lim_{n\to\infty}\int_X f(x)r_n(x)d\nu(x) = \int_X f(x)r(x)d\nu(x)$$

are valid.

If we approximate any measurable function by Baire functions, then we get

$$\int_X f(x)d\mu(x) = \int_X f(x)r(x)d\nu(x).$$

and the theorem is proved. □

Corollary 4.4 (Kakutani) Let, for an arbitrary $k \in N$, μ_k be the canonical Gaussian probability Borel measure defined on $\mathbf{R}$ and having the mean 0 and the variance 1. Let $X = \mathbf{R}^N$ and $\mu = \prod_{k\in N}\mu_k$ be a canonical Gaussian Borel probability measure in $\mathbf{R}^N$. Then, a vector space Q_μ of all admissible (in the sense of quasiinvariance) translations for the measure μ coincides with ℓ_2.

Proof. Let us denote by μ_a the measure defined by

$$(\forall B)(B \in B(\mathbf{R}^N) \to \mu_a(B) = \mu(B+a)),$$

where $a = (a_k)_{k\in N} \in \mathbf{R}^N$.

Obviously, we have

$$\mu_a = \prod_{k\in N}\nu_k,$$

where

$$(\forall B)(B \in B(\mathbf{R}) \to \nu_k(B) = \int_B \frac{1}{\sqrt{2\pi}}e^{-\frac{(t_k-a_k)^2}{2}}dt).$$

Indeed, we get

$$\alpha_k = \int_R \sqrt{e^{-\frac{t_k^2}{2}+\frac{(t_k-a_k)^2}{2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t_k-a_k)^2}{2}}dt_k =$$

$$= \int_R \frac{1}{\sqrt{2\pi}}e^{-\frac{(t_k-\frac{a_k}{2})^2}{2}-\frac{a_k^2}{8}}dt_k = e^{-\frac{a_k^2}{8}}.$$

It is not difficult to check that

$$\prod_{k\in N}\alpha_k = e^{-\frac{\sum_{k\in N}a_k^2}{8}}$$

is divergent to zero if and only if the series

$$\sum_{k\in N} a_k^2$$

is convergent in the usual sense , i.e., $(a_k)_{k\in N} \in \ell_2$. $\square$

Corollary 4.5 Let ν_k and μ_k be two Gaussian probability measures with the density functions

$$\frac{d\nu_k(x_k)}{dx_k} = \frac{1}{\sigma_k\sqrt{2\pi}} e^{-\frac{(x_k-\gamma_k)^2}{2\sigma_k^2}}$$

and

$$\frac{d\mu_k(x_k)}{dx_k} = \frac{1}{\lambda_k\sqrt{2\pi}} e^{-\frac{(x_k-\beta_k)^2}{2\lambda_k^2}},$$

respectively, for $k \in N$.

Let us put $X = \mathbf{R}^\infty$.

In this case

$$\rho_k(x_k) = \frac{\sigma_k}{\lambda_k} e^{-\frac{1}{2\sigma_k^2\lambda_k^2}[(x_k-\beta_k)^2\sigma_k^2-(x_k-\gamma_k)^2\lambda_k^2]}.$$

Simple calculations show us that

$$\alpha_k = \frac{1}{\sqrt{\sigma_k\lambda_k}A}\cdot C\cdot e^D,$$

where

$$A = \sqrt{\sigma_k^2+\lambda_k^2},\ C = \sqrt{2}\sigma_k\cdot\lambda_k,\ D = -\frac{(\beta_k-\gamma_k)^2}{\sigma_k^2+\lambda_k^2}.$$

Eventually, we get

$$\alpha_k = \sqrt{\frac{2\sigma_k\lambda_k}{\sigma_k^2+\lambda_k^2}}\cdot e^{-\frac{(\beta_k-\gamma_k)^2}{4(\sigma_k^2+\lambda_k^2)}}.$$

By using Theorem 4.3 we conclude that the product-measures $\nu = \prod_{k\in N}\nu_k$ and $\mu = \prod_{k\in N}\mu_k$ are equivalent if and only if the infinite product

$$\prod_{k\in N}\sqrt{\frac{2\sigma_k\lambda_k}{\sigma_k^2+\lambda_k^2}}\cdot e^{-\frac{(\beta_k-\gamma_k)^2}{4(\sigma_k^2+\lambda_k^2)}}$$

is divergent to zero.

Corollary 4.6 (Pitcher) Let μ be the canonical Gaussian measure defined on $\mathbf{R}^N$. Let

$$U : \mathbf{R}^N \to \mathbf{R}^N$$

be a transformation of $\mathbf{R}^N$ into itself defined by

$$U((x_i)_{i\in N}) = (\lambda_i x_i)_{i\in N},$$

where $(x_i)_{i\in N} \in \mathbf{R}^N$ and $\lambda_i > 0$. Then, according to Corollary 4.5, the Gaussian measure μ is quasiinvariant under the transformation U if and only if the product

$$\prod_{i\in N}\sqrt{\frac{2\lambda_i}{1+\lambda_i^2}}$$

is divergent to zero.

Example 4.2 Let V be a transformation defined by

$$V((x_k)_{k\in N}) = (\lambda_k x_k)_{k\in N},$$

where

$$\lambda_k = (1+\sqrt{1-e^{2\beta_k}})e^{-\beta_k},$$

$$(\forall k)(k\in N \rightarrow \beta_k \leq 0)\ \&\ (\beta_k)_{k\in N} \in \ell_1.$$

It is clear that

$$\prod_{k=1}^{\infty}\sqrt{\frac{2\lambda_k}{1+\lambda_k^2}} = \prod_{k=1}^{\infty} e^{\beta_k} = e^{\sum_{k\in N}\beta_k}$$

and, according to Corollary 4.6, the measure μ is quasiinvariant under the transformation V.

The following result is valid.

Corollary 4.7 *Let α be an arbitrary infinite parameter set, and μ be the canonical Gaussian Borel measure defined in $\mathbf{R}^\alpha$. Then the vector space Q_μ of all admissible (in ther sense of quasiinvariance) translations for the measure μ coincides with $\ell_2(\alpha)$, where*

$$\ell_2(\alpha) = \{(x_i)_{i\in\alpha} \in \mathbf{R}^\alpha\ \&\ \sum_{i\in\alpha} x_i^2 < \infty\}.$$

Proof. Let us denote by μ_J the canonical Gaussian probability measure defined on $\mathbf{R}^J$ ($J \subseteq \alpha$). If we denote by $\mu_{\alpha\setminus J}$ the canonical Gaussian Borel probability measure defined on $\mathbf{R}^{\alpha\setminus J}$, then, using Lemma 4.4, we get

$$(\forall J)(J \subseteq \alpha\ \&\ \mathrm{Card(J)} \leq \aleph_0 \rightarrow \mu = \mu_J \times \mu_{\alpha\setminus J}).$$

Ad hoc μ_J is an ℓ_2-quasiinvariant measure, μ is $\ell_2(J)\times I_{\alpha\setminus J}$ quasiinvariant for all $J \subseteq \alpha$ with $\mathrm{card}(J) = \aleph_0$, where $I_{\alpha\setminus J}$ is a zero of $\mathbf{R}^{\alpha\setminus J}$, which can be identified with the degenerate translation of $\mathbf{R}^{\alpha\setminus J}$.

It is clear that

$$\ell_2(\alpha) \subseteq \cup_{J\subseteq\alpha,\mathrm{card}(J)=\aleph_0}\ell_2(J)\times I^{\alpha\setminus J}.$$

Let $(x_i)_{i\in\alpha} \in Q_\mu$. Let us assume the contrary and let $(x_i)_{i\in\alpha} \notin \ell_2(\alpha)$. Then

$$\sum_{i\in\alpha} x_i^2 = +\infty.$$

It is clear that we can indicate a countable subspace $J_0 = (i_m)_{m \in N} \subseteq \alpha$ such that

$$\sum_{i \in J_0} x_i^2 = +\infty.$$

By Corollary 4.4, we can choose a Borel set $B_0 \subseteq \mathbf{R}^{J_0}$ such that

$$\mu_{J_0}(B_0) > 0 \ \& \ \mu_{J_0}(B_0 + (x_i)_{i \in J_0}) = 0.$$

Then from the relations

$$\mu(B_0 \times \mathbf{R}^{\alpha \setminus J_0}) = \mu_{J_0}(B_0) > 0 \ \&$$

$$\mu(B_0 \times \mathbf{R}^{\alpha \setminus J_0} + (x_i)_{i \in \alpha}) = \mu((B_0 + (x_i)_{i \in J_0}) \times (\mathbf{R}^{\alpha \setminus J_0} + (x_i)_{i \in \alpha \setminus J_0}) =$$

$$= \mu((B_0 + (x_i)_{i \in J_0}) \times \mathbf{R}^{\alpha \setminus J_0}) = \mu_{J_0}(B_0 + (x_i)_{i \in J_0}) = 0$$

we get a contradiction to the condition

$$(x_i)_{i \in I} \in Q_\mu,$$

and Corollary 4.7 is proved. □

Corollary 4.8 *Let $a_i \in \mathbf{R}_i$ $(i \in \alpha)$. Denote by $p_i^{(a_i)}$ the measure defined by*

$$(\forall B)(B \in B(\mathbf{R}) \rightarrow p_i^{(a_i)}(B) = p_i(B + a_i)).$$

If we put

$$\rho_i^{(a_i)} = \frac{dp_i^{(a_i)}}{dp_i},$$

and

$$\alpha_i = \int_R \sqrt{\rho_i^{(a_i)}(x_i)} dp_i(x_i),$$

then, for Q_ν, the following representation

$$Q_\nu = \{(a_i)_{i \in \alpha} : (\forall J)(J \subseteq \alpha \ \& \ card(J) \leq \aleph_0 \rightarrow \prod_{i \in J} \alpha_i \textit{ is divergent to zero})\},$$

is valid, where $\nu = \prod_{i \in \alpha} p_i$.

A characterization of all transformations of $\mathbf{R}^\alpha$ generated by infinite diagonal matrices under which the canonical Gaussian Borel probability measures are quasiinvariant, is presented in the following corollary.

Corollary 4.9 *Let α be an arbitrary set of parameters. Then the canonical Gaussian Borel measure μ defined on $\mathbf{R}^\alpha$ is quasiinvariant under the transformation U having the form*

$$U((x_i)_{i \in \alpha}) = (\lambda_i x_i)_{i \in \alpha}((x_i)_{i \in \alpha} \in R^\alpha)$$

if and only if there exists $\alpha_0 \subseteq \alpha$ *with card* $(\alpha_0) \leq \aleph_0$ *such that the product*

$$\prod_{i\in\alpha_0}\sqrt{\frac{2\lambda_i}{1+\lambda_i^2}}$$

is divergent to zero and

$$(\forall i)(i \in \alpha \setminus \alpha_0 \to \lambda_i = 1).$$

Proof.

Sufficiency. Denote by $\overline{U}$ the transformation defined by

$$\overline{U}((x_i)_{i\in\alpha_0}) = (\lambda_i x_i)_{i\in\alpha_0}.$$

Analogously, denote by $\overline{V}$ the identity transformation of $\mathbf{R}^{\alpha\setminus\alpha_0}$ into itself. Denote by μ_{α_0} and $\mu_{\alpha\setminus\alpha_0}$ the canonical Gaussian Borel measures defined on $\mathbf{R}^{\alpha}$ and $\mathbf{R}^{\alpha\setminus\alpha_0}$, respectively. It is clear that

$$\mu = \mu_{\alpha_0} \times \mu_{\alpha\setminus\alpha_0}.$$

According Corollary 4.6 we conclude that μ_{α_0} is quasiinvariant under the transformation $\overline{U}$. The measure $\mu_{\alpha\setminus\alpha_0}$ can be considered as quasiinvariant under the transformation $\overline{V}$. Hence the measure μ is quasiinvariant under the transformation $\overline{U}\times\overline{V}$ and the sufficiency is proved.

Necessity. Let μ be quasiinvariant under the transformation U. It is clear that for an arbitrary countable parameter set $\alpha_0 \subseteq \alpha$, the product

$$\prod_{i\in\alpha_0}\sqrt{\frac{2\lambda_i}{1+\lambda_i^2}}$$

is divergent to zero.

Indeed, if we assume that there exists $\alpha_0 \subseteq \alpha$ such that $\prod_{i\in\alpha_0}\sqrt{\frac{2\lambda_i}{1+\lambda_i^2}}$ is convergent to zero, then by Corollary 4.6 the canonical Gaussian Borel measure μ_{α_0} is not quasiinvariant under the transformation $\overline{U} : \mathbf{R}^{\alpha_0} \to \mathbf{R}^{\alpha_0}$, where

$$\overline{U}((x_i)_{i\in\alpha_0}) = (\lambda_i x_i)_{i\in\alpha_0}.$$

The latter relation means that there exists $B_0 \in B(\mathbf{R}^{\alpha_0})$ such that

$$\mu_{\alpha_0}(B_0) > 0 \ \&\ \mu_{\alpha_0}(\overline{U}(B_0)) = 0.$$

Let us define a transformation $\overline{V} : \mathbf{R}^{\alpha\setminus\alpha_0} \to \mathbf{R}^{\alpha\setminus\alpha_0}$ by

$$\overline{V}((x_i)_{i\in\alpha\setminus\alpha_0}) = (\lambda_i x_i)_{i\in\alpha\setminus\alpha_0}.$$

It is clear that

$$\mu(B_0 \times \mathbf{R}^{\alpha\setminus\alpha_0}) = \mu_{\alpha_0}(B_0) \times \mu_{\alpha\setminus\alpha_0}(\mathbf{R}^{\alpha\setminus\alpha_0}).$$

On the other hand, we have

$$\mu(U(B_0 \times \mathbf{R}^{\alpha\setminus\alpha_0})) = \mu_{\alpha_0}(\overline{U}(B_0) \times \overline{V}(\mathbf{R}^{\alpha\setminus\alpha_0})) =$$

$$= \mu_{\alpha_0}(\overline{U}(B_0)) \times \mu_{\alpha\setminus\alpha_0}(\overline{V}(\mathbf{R}^{\alpha\setminus\alpha_0})) =$$

$$= \mu_{\alpha_0}(\overline{U}(B_0)) \times \mu_{\alpha\setminus\alpha_0}(\mathbf{R}^{\alpha\setminus\alpha_0}) = \mu_{\alpha_0}(\overline{U}(B_0)) = 0$$

and we get a contradiction with the condition of quasiinvariance of the measure μ under the transformation U.

Let us assume that

$$\mathrm{card}(\{i : \lambda_i \neq 1\}) > \aleph_0.$$

Obviously, we get

$$\mathrm{card}(\{i : 0 < \sqrt{\frac{2\lambda_i}{1+\lambda_i^2}} < 1\}) \geq \omega_1.$$

Now we can easily conclude that

$$(\exists a)(0 < a < 1 \rightarrow \mathrm{card}(\{i : 0 < \sqrt{\frac{2\lambda_i}{1+\lambda_i^2}} < a < 1\}) \geq \omega_1).$$

If we consider only a countable infinite subset $\alpha_0 \subseteq \alpha$ such that

$$(\forall i)(i \in \alpha_0 \rightarrow 0 \leq \sqrt{\frac{2\lambda_i}{1+\lambda_i^2}} < a < 1),$$

then, using Corollary 4.6, we obtain that μ_{α_0} is not quasiinvariant under the transformation $\overline{U} : \mathbf{R}^{\alpha_0} \rightarrow \mathbf{R}^{\alpha_0}$. Hence the measure μ is not quasiinvariant under the transformation U and Corollary 4.9 is proved. □

Now we discuss a different point of view to describe a structure of Gaussian Baire measures in $\mathbf{R}^\alpha$ for arbitrary nonempty parameter set α. Applying some information on a given Gaussian process, we will construct a new Gaussian process such that its corresponding Gaussian Baire measure coincides with the Gaussian Baire measure generated by the early given process. This construction allows us to extend some results obtained in [19],[48].

We continue our discussion with some notions from the theory of random processes.

We recall that a Baire probability measure μ on $\mathbf{R}^\alpha$ is called a Gaussian if its all finite-dimensional distributions are Gaussian.

Let $(\eta_i)_{i\in\alpha}$ be an arbitrary process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The measure μ, defined by

$$(\forall X)(X \in Ba(\mathbf{R}^\alpha) \rightarrow \mu(X) = \mathcal{P}(\{\omega : (\eta_i(\omega))_{i\in\alpha} \in X\}))$$

is called a Baire measure on $\mathbf{R}^\alpha$ generated by the process $(\eta_i)_{i\in\alpha}$.

Remark 4.8 Note that it is not possible, in general, to define a Borel measure on R^α(for $\mathrm{card}(\alpha) > \omega$) for the process $(\eta_i)_{i\in\alpha}$ by above-mentioned formula, because the set

$$\{\omega : (\eta_i(\omega))_{i\in\alpha} \in X\}$$

is not always an element of $\mathcal{F}$ for arbitrary $X \in B(R^\alpha)$.

In the sequel we shall need some auxiliary propositions.

Lemma 4.7 *An arbitrary Baire Gaussian measure μ on $\mathbf{R}^{\alpha}$ can be represented as a Baire measure on $\mathbf{R}^{\alpha}$ generated by some Gaussian process $(\xi_i)_{i\in\alpha}$.*

Proof. The proof of Lemma 4.7 can be obtained obviously if we set:
1) $(\Omega,\mathcal{F},\mathcal{P})=(\mathbf{R}^{\alpha},Ba(\mathbf{R}^{\alpha}),\mu)$,
2) $(\forall i)(i\in\alpha\rightarrow\eta_i=Pr_i)$.

Now, let $(\eta_i)_{i\in\alpha}$ be an arbitrary Gaussian process defined on the probability space $(\Omega,\mathcal{F},\mathcal{P})$ and let $(a_i)_{i\in\alpha}=(\int_{\Omega}\xi_i d\mathcal{P})_{i\in\alpha}\in\mathbf{R}^{\alpha}$ be a mean of the given process.
Let us consider a process $(\xi_i)_{i\in\alpha}$ defined by

$$(\forall i)(i\in\alpha\rightarrow\xi_i=\eta_i-a_i).$$

Following [161](cf. Chapter II, Paragraph 13) a linear vector space H, consisting of values of the kind $\sum_{i=1}^{n}c_{i_k}\xi_{i_k}$ and of its limits in the sense of $\mathcal{L}_2(\Omega,\mathcal{F},\mathcal{P})$, generates a Gaussian system. Hence, H can be considered as a Hilbert space with a usual scalar product $<.,.>$, defined by

$$(\forall f)(\forall g)(f\in H\ \&\ g\in H\ \rightarrow<f,g>=\int_{\Omega}fgd\mathcal{P}\ ).$$

As usual, H is called a Hilbert space generated by the Gaussian process $(\xi_i)_{i\in\alpha}$. Let $(\gamma_j)_{j\in J}$ be an orthonormal basis in H (cf. [154],Theorem II.5). Then, it is clear that

$$(\forall i)(i\in\alpha\rightarrow\xi_i=\sum_{j\in J}<\xi_i,\gamma_j>\gamma_j).$$

A matrix $\mathcal{A}=(<\xi_i,\gamma_j>)_{i\in\alpha,j\in J}$ is called a matrix of representation of the process $(\eta_i-a_i)_{i\in\alpha}$ in the basis $(\gamma_j)_{j\in J}$. A main property of the matrix $\mathcal{A}$ is the following:

$$(\forall i)(i\in\alpha\rightarrow\sum_{j\in J}<\xi_i,\gamma_j>^2<\infty).$$

It is well known that if $(X_n)_{n\in N}$ is a sequence of independent real-valued random values, defined on $(\Omega,\mathcal{F},\mathcal{P})$, then for a series $\sum_{n\in N}X_n$ the notions of convergence almost everywhere, convergence in the probability and convergence in the distribution are equivalent(cf. [64],p.43,Theorem 1.9.). It is reasonable to note that if $(X_n)_{n\in N}$ is a sequence of independent functions such that $\int_{\Omega}X_n d\mathcal{P}=0$ and $\sum_{n\in N}\int_{\Omega}X_n^2 d\mathcal{P}<\infty$, then the series $\sum_{n=1}^{\infty}X_n$ converges a.e. with respect to the measure $\mathcal{P}$ (see, e.g., [54],p.197,Theorem B). We know also that if the series

$$\sum_{k=1}^{\infty}<\xi,\gamma_k>\gamma_k$$

converges (in the sense of convergence of Hilbert space H) to $\xi\in H$ and f is on-to-one function on N to itself, then the series $\sum_{n=1}^{\infty}<\xi,\gamma_{f(n)}>\gamma_{f(n)}$ also converges to ξ in above mentioned sense. This property can be called a property of σ-commutativity of the series $\sum_{k=1}^{\infty}<\xi,\gamma_k>\gamma_k$ in H. Following [54](cf. p.197,Theorem B) the series $\sum_{k=1}^{\infty}<\xi,\gamma_k>\gamma_k$ has also the property of σ-commutativity a.e. with respect to the measure $\mathcal{P}$, i.e., for arbitrary fixed permutation f of N, the condition

$$\mathcal{P}(\{\omega:\lim_{n\rightarrow\infty}(\sum_{k=1}^{n}<\xi,\gamma_k>\gamma_k(\omega)-\sum_{k=1}^{n}<\xi,\gamma_{f(k)}>\gamma_{f(k)}(\omega))=0\ \&$$

$$\& \lim_{n\to\infty}\sum_{k=1}^{n} < \xi,\gamma_k > \gamma_k(\omega) \text{ is finite}\}) = 1$$

holds.

Let us consider a Baire Gaussian measure $\mu_\alpha^{G(A)}$, defined by

$$(\forall X)(X \in Ba(\mathbf{R}^\alpha) \rightarrow \mu_\alpha^{G(A)} = \mathcal{P}(\{\omega : (\xi_i(\omega))_{i\in\alpha} \in X\})).$$

It is reasonable to note that the measure $\mu_\alpha^{G(A)}$ can be obtained by the following construction: let μ_J^G be a canonical Baire Gaussian measure defined on $\mathbf{R}^J$. For an arbitrary $i \in \alpha$ we set

$$J_i = \{j : j \in J \;\& \; < \xi_i,\gamma_j > \neq 0\}.$$

It is clear that $\mathrm{card}(J_i) \leq \aleph_0$. Let $g_i : N \rightarrow J$ be a such injective function that $J_i \subseteq g_i(N)$. We set $A_i((x_j)_{j\in J}) = \sum_{n=1}^{\infty} < \xi_i,\gamma_{g_i(n)} > x_{g_i(n)}$, if this series converges in usual sense to the finite number, and $=$ 0, in all other cases.

As $(Pr_j)_{j\in J}$ is the sequence of independent Gaussian random variables defined on $(\mathbf{R}^J, Ba(\mathbf{R}^J), \mu_J^G)$, for every fixed $i \in \alpha$ we have

$$\mu_J^G(\{(x_s)_{s\in J} : \sum_{n=1}^{\infty} < \xi_i,\gamma_{g_i(n)} > Pr_{g_i(n)}((x_s)_{s\in J}) \text{ converges in usual sense}$$

$$\text{to the finite number}\}) = 1.$$

Hence,

$$\mu_J^G(\{(x_j)_{j\in J} : \sum_{n=1}^{\infty} < \xi_i,\gamma_{g_i(n)} > x_{g_i(n)} \;\text{ converges in usual sense}$$

$$\text{to the finite number}\}) = 1.$$

We set

$$A((x_j)_{j\in J}) = (A_i((x_j)_{j\in J}))_{i\in\alpha}.$$

One can easily check that the operator A is measurable in the following sense:

$$(\forall X)(X \in Ba(\mathbf{R}^\alpha) \rightarrow A^{-1}(X) \in Ba(\mathbf{R}^J)).$$

Using the well-known Kolmogoroff theorem(cf.[161],Chapter II,Paragraph 2), it is not difficult to prove the validity of the following equality:

$$(\forall X)(X \in Ba(\mathbf{R}^\alpha) \rightarrow \mu_\alpha^{G(A)}(X) = \mu_J^G(A^{-1}(X))).$$

Let μ be a Gaussian Baire measure on $\mathbf{R}^\alpha$ generated by the process $(\eta_i)_{i\in\alpha}$. Then we have

$$(\forall X)(X \in Ba(\mathbf{R}^\alpha) \rightarrow \mu(X) = P(\{\omega : (\eta_i(\omega))_{i\in\alpha} \in X\}) =$$

$$P(\{\omega : (\xi_i(\omega))_{i\in\alpha} \in X - (a_i)_{i\in\alpha}\}) = \mu_\alpha^{G(A)}(X - (a_i)_{i\in\alpha}) =$$

$$\mu_J^G(A^{-1}(X - (a_i)_{i\in\alpha})).$$

Thus, we have established the main result of the present chapter which gives a structure of Gaussian Baire measures on $\mathbf{R}^{\alpha}$ for arbitrary parameter set α.

Theorem 4.4 *For arbitrary Gaussian Baire measure μ on $\mathbf{R}^{\alpha}$ generated by any Gaussian process $(\eta_i)_{i\in\alpha}$ there exists a parameter set J such that μ can be represented by the following formula*

$$(\forall X)(X \in Ba(\mathbf{R}^{\alpha}) \to \mu(X) = \mu_J^G(A^{-1}(X-(a_i)_{i\in\alpha}))),$$

where J is cardinality of the basis $(\gamma_j)_{j\in J}$ of the Hilbert space generated by the process $(\xi_i)_{i\in\alpha} = (\eta_i - a_i)_{i\in\alpha}$, $A : \mathbf{R}^J \to \mathbf{R}^{\alpha}$ is the operator generated by the representation matrix $\mathcal{A}$ of the process $(\xi_i)_{i\in\alpha}$ in the basis $(\gamma_j)_{j\in J}$, and $a = (a_i)_{i\in\alpha}$ is the mean of the process $(\eta_i)_{i\in\alpha}$.

Remark 4.9 In the sequel for the Gaussian Baire measure μ, described in Theorem 4.4 , we shall use a notation $\mu_{\alpha}^{(G(A),a)}$. For canonical Gaussian Baire measure on $\mathbf{R}^{\alpha}$ we preserve a notation μ_{α}^{G} which coincides with above-setting notation when A is the identical operator in $\mathbf{R}^{\alpha}$ and the mean a is equal to zero of $\mathbf{R}^{\alpha}$. For Gaussian measures with the representation matrix $\mathcal{A}$ and the mean zero of R^{α} we shall apply a notation $\mu_{\alpha}^{G(A)}$.

The following result is a consequence of Theorem 4.4 and Corollary 4.7.

Theorem 4.5 *The Gaussian Baire measure $\mu_{\alpha}^{(G(A),a)}$ is $A(\ell_2(J))$- quasiinvariant.*

Proof. We set
$M_i = \{(x_j)_{j\in J} : \sum_{n=1}^{\infty} < \xi_i, \gamma_{g_i(n)} > x_{g_i(n)}$ converges in usual sense to the finite number $\}$
and

$$(\forall \tau)(\tau \subseteq \alpha \ \to M_{\tau} = \cap_{i\in\tau} M_i).$$

Then, for an arbitrary $\tau \subseteq \alpha$ with $\mathrm{card}(\tau) \le \omega$, the following conditions are fulfilled:
(i) $M_{\tau} \in Ba(\mathbf{R}^J)$ & $\mu_J^G(M_{\tau}) = 1$,
(ii) M_{τ} is a vector subspace of $\mathbf{R}^J$,
(iii) $(\forall i)(i \in \tau \to A_i$ is a linear operator on M_{τ} & $(\forall Y)(Y \in B(\mathbf{R}) \to A_i^{-1}(Y) \in Ba(\mathbf{R}^{\alpha}) \cap M_{\tau}))$.

Let $B = B^{\tau_0} \times \mathbf{R}^{\alpha\setminus\tau_0}$ and $h = A((h_j^{(0)})_{j\in J})$, where $\tau_0 \subset \alpha, \mathrm{card}(\tau_0) \le \omega$, $B^{\tau_0} \in Ba(R^{\tau_0})$ and $(h_j^{(0)})_{j\in J} \in \ell_2(J)$. Let us prove the validity of the following equality

$$M_{\tau_0} \cap A^{-1}(B+h) = M_{\tau_0} \cap (A^{-1}(B)+h).$$

Let $(x_j)_{j\in J} \in M_{\tau_0} \cap A^{-1}(B+h)$. Then:
1) $(\forall i)(i \in \tau_0 \to \sum_{n=1}^{\infty} < \xi_i, \gamma_{g_i(n)} > x_{g_i(n)}$ converges in usual sense to the finite number);
2) $A((x_j)_{j\in J}) \in B^{\tau_0} \times \mathbf{R}^{\alpha\setminus\tau_0} + A((h_j^{(0)})_{j\in J})$.
Hence,

$$A((x_j)_{j\in J}) - A((h_j^{(0)})_{j\in J}) \in B^{\tau_0} \times \mathbf{R}^{\alpha\setminus\tau_0} \to (A_i((x_j)_{j\in J}))_{i\in\tau_0} -$$

$$(A_i((h_j^{(0)})_{j\in J}))_{i\in\tau_0} \in B^{\tau_0} \to (A_i((x_j)_{j\in J} - (h_j^{(0)})_{j\in J}))_{i\in\tau_0} \in B^{\tau_0} \to$$

$$(A_i((x_j - h_j^{(0)})_{j\in J}))_{i\in\tau_0} \in B^{\tau_0} \to A((x_j - h_j^{(0)})_{j\in J}) \in B^{\tau_0} \times \mathbf{R}^{\alpha\setminus\tau_0} \to$$

$$A((x_j)_{j\in J}-(h_j^{(0)})_{j\in J})\in B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0}\to(x_j)_{j\in J}-(h_j^{(0)})_{j\in J}\in A^{-1}(B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0})\to$$
$$(x_j)_{j\in J}\in A^{-1}(B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0})+(h_j^{(0)})_{j\in J}.$$

Hence, the validity of the inclusion

$$M_{\tau_0}\cap A^{-1}(B+h)\subseteq M_{\tau_0}\cap(A^{-1}(B)+h)$$

is proved.

Let us show the validity of the converse inclusion.

Let $(x_j)_{j\in J}\in M_{\tau_0}\cap(A^{-1}(B)+h)$. Then:

1) $(\forall i)(i\in\tau_0\to\sum_{n=1}^{\infty}<\xi_i,\gamma_{g_i(n)}>x_{g_i(n)}$ converges in usual sense to the finite number);

2) $(x_j)_{j\in J}\in A^{-1}(B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0})+(h_j^{(0)})_{j\in J}$.

Hence,

$$(x_j-h_j^{(0)})_{j\in J}\in A^{-1}(B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0})\to A((x_j-h_j^{(0)})_{j\in J})\in B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0}\to$$
$$(A_i((x_j-h_j^{(0)})_{j\in J}))_{i\in\tau_0}\in B^{\tau_0}\to(A_i((x_j)_{j\in J}))_{i\in\tau_0}-(A_i((x_j)_{j\in J}))_{i\in\tau_0}\in B^{\tau_0}\to$$
$$(A_i((x_j)_{j\in J}))_{i\in\alpha}-(A_i((h_j^{(0)})_{j\in J}))_{i\in\alpha}\in B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0}\to(A_i((x_j)_{j\in J}))_{i\in\alpha}\in$$
$$B^{\tau_0}\times\mathbf{R}^{\alpha\setminus\tau_0}+(A_i((h_j^{(0)})_{j\in J}))_{i\in\alpha}\to(x_j)_{j\in J}\in A^{-1}(B+h).$$

Hence, the validity of the equality

$$M_{\tau_0}\cap A^{-1}(B+h)=M_{\tau_0}\cap(A^{-1}(B)+h)$$

is proved.

Now let us prove $A(\ell_2(J))$-quasiinvariance of the measure $\mu_\alpha^{G(A)}$.

Let $\mu_\alpha^{G(A)}(B)>0$, where $B=B^\tau\times\mathbf{R}^{\alpha\setminus\tau}$, $B^\tau\in B(\mathbf{R}^\tau)$, $\tau\subseteq\alpha$ & $\mathrm{card}(\tau)\le\omega$. Let $h_0\in\ell_2(J)$ and $h=A(h_0)$. We have

$$\mu_\alpha^{G(A)}(B+h)=\mu_J^G(A^{-1}(B+h))=\mu_J^G(M_\tau\cap A^{-1}(B+h))=$$
$$=\mu_J^G(M_\tau\cap(A^{-1}(B)+h_0))=\mu_J^G(A^{-1}(B)+h_0).$$

Note that the condition

$$\mu_\alpha^{G(A)}(B)>0$$

implies

$$\mu_J^G(A^{-1}(B))>0.$$

From the $\ell_2(J)$-quasiinvariance of the measure μ_J^G, we have

$$\mu_J^G(A^{-1}(B)+h_0)>0.$$

But last relation means that

$$\mu_\alpha^{G(A)}(B+h)>0.$$

Hence, the measure $\mu_\alpha^{G(A)}$ is $A(\ell_2(\alpha))$-quasiinvariant. Consequently,

$$\mu_\alpha^{G(A)}(X-a)>0$$

if and only if

$$\mu_\alpha^{G(A)}(X-a+h)>0$$

for all $h\in A(\ell_2(\alpha))$, but last relation means that $\mu_\alpha^{(G(A),a)}$ is $A(\ell_2(\alpha))$-quasiinvariant, because

$$(\forall X)(X\in Ba(\mathbf{R}^\alpha)\to\mu_\alpha^{(G(A),a)}(X)=\mu_\alpha^{G(A)}(X-a)).$$

Theorem 4.5 is proved. $\square$

Now let us consider some corollaries of Theorem 4.5.

We recall the reader that a measure μ, defined on an infinite-measurable separable Hilbert space $(H,\mathcal{F})$, is called a Gaussian if an arbitrary continuous linear functional $\ell_z(x)=(z,x)(z,x\in H)$ is a normally distributed random value (cf.[48], Chapter **v**, Paragraph 6). Note that $(\xi_z)_{z\in H}$, defined by $\xi_z=\ell_z$ for $z\in H$, is a Gaussian process on $(H,\mathcal{F},\mu)$. It is clear also that the Baire measure λ on $\mathbf{R}^H$, generated by the process $(\xi_z)_{z\in H}$, is a Gaussian Baire measure. As $a(z)=\int_H(z,x)\mu(d(x))$ and $b(z)=\int_H(z,x)^2\mu(d(x))-(\int_H(z,x)\mu(d(x)))^2$ are polylinear momentum forms in H (cf.[48], Chapter **v**,Paragraph 5), there exists $a\in H$ and a bounded by a norm symmetric nondegenerate linear operator B such that $a(z)=(a,z)$ and $b(z)=(Bz,z)$. As usual a and B are called a mean and a correlation operator of the measure μ, respectively. Assume that a mean $(a(z))_{z\in H}$ of the measure μ is equal to zero. Let $(\lambda_k)_{k\in N}$ be a proper numbers of the operator B and $(e_k)_{k\in N}$ be an orthonormal basis of H generated by proper vectors of B such that $Be_k=\lambda_k e_k$ for $k\in N$.

Corollary 4.10 *The measure λ is $A(\ell_2)$-quasiinvariant, where*

$$A(\ell_2)=\{(b_z)_{z\in H}:(\exists(\alpha_k)_{k\in N}\in\ell_2\ \&\ (\forall z)(z\in H\to b_z=\sum_{k\in N}\sqrt{\lambda_k}\alpha_k(z,e_k))\}.$$

Proof. Let us consider a Hilbert space H' generated by $(\xi_z)_{z\in H}$. Let us show that the family of linear functionals $(\Psi_k)_{k\in N}=(\ell_{\frac{e_k}{\sqrt{\lambda_k}}})_{k\in N}$ is a basis in H'. In this direction it is sufficient to show that $(\Psi_k)_{k\in N}$ is orthonormal and for arbitrary linear functional ℓ_z there exists a sequence of real numbers $(c_k)_{k\in N}\in\ell_2$ such that $(\sum_{k=1}^n c_k\Psi_k)_{n\in N}$ tends to ℓ_z in the sense of $\mathcal{L}_2(H,\mathcal{F},\mu)$.

Indeed:

1) The orthonormality can be proved as follows:

$$<\Psi_k,\Psi_m>=\int_H\Psi_k(x)\Psi_m(x)\mu(d(x))=\int_H(x,\frac{e_k}{\sqrt{\lambda_k}})(x,\frac{e_m}{\sqrt{\lambda_m}})\mu(d(x))=$$

$$(B\frac{e_k}{\sqrt{\lambda_k}},\frac{e_m}{\sqrt{\lambda_m}})=$$

$$\frac{1}{\sqrt{\lambda_k\lambda_m}}(\lambda_k e_k,e_m)=\frac{\sqrt{\lambda_k}}{\sqrt{\lambda_m}}(e_k,e_m)=\left\{\begin{array}{ll}0, & \text{if } k\neq m,\\ 1, & \text{if } k=m.\end{array}\right\}$$

2) Note that

$$< f_z, \Psi_k >= \int_H (x,z)(x, \frac{e_k}{\sqrt{\lambda_k}})\mu(d(x)) = (Bz, \frac{e_k}{\sqrt{\lambda_k}}) =$$

$$= (B(\sum_{m\in N}(z,e_m)e_m), \frac{e_k}{\sqrt{\lambda_k}}) = (\sum_{m\in N}(z,e_m)Be_m, \frac{e_k}{\sqrt{\lambda_k}}) =$$

$$= (\sum_{m\in N}(z,e_m)\lambda_m e_m, \frac{e_k}{\sqrt{\lambda_k}}) = \sqrt{\lambda_k}(z,e_k).$$

Set $c_k = \sqrt{\lambda_k}(z,e_k)$ for $k \in N$. We have

$$\lim_{n\to\infty} \Phi_{\sum_{k=1}^n c_k\Psi_k}(t) = \lim_{n\to\infty}\int_H e^{it\sum_{k=1}^n c_k\Psi_k(x)}\mu(d(x)) =$$

$$= \lim_{n\to\infty}\prod_{k=1}^n \int_H e^{itc_k\Psi_k(x)}\mu(d(x)) = \lim_{n\to\infty}\prod_{k=1}^n e^{-\frac{t^2}{2}\lambda_k(z,e_k)^2} =$$

$$= e^{-\frac{t^2}{2}\sum_{k\in N}\lambda_k(z,e_k)^2}.$$

On the other hand, we have

$$\Phi_{\ell_z}(t) = Ee^{it\ell_z} = \int_H e^{it(z,x)}\mu(d(x)) = e^{-\frac{t^2}{2}(Bz,z)} =$$

$$= e^{-\frac{t^2}{2}(B\sum_{k\in N}c_ke_k, \sum_{k\in N}c_ke_k)} = e^{-\frac{t^2}{2}\sum_{k\in N}\lambda_k(z,e_k)^2}.$$

Hence, following [161](cf. Chapter III,Paragraph 3), the series $(c_k\Psi_k)_{k\in N}$ converges in the distribution to ℓ_z. As $(c_k\Psi_k)_{k\in N}$ is a sequence of independent Gaussian random values, for series $\sum_{k\in N} c_k\Psi_k$ the notions of convergence in the distribution and convergence in the sense of $\mathcal{L}_2(H,\mathcal{F},\mu)$ are equivalent (cf.[64],p.43, Theorem 1.9).

We set $\mathcal{A} = (\sqrt{\lambda_k}(z,e_k))_{H,N}$. According to Theorem 4.5, the measure $\mu_H^{G(A)}$ is $A(\ell_2)$-quasiinvariant, where

$$A(\ell_2) = \{(b_z)_{z\in H} : (\exists(\alpha_k)_{k\in N} \in \ell_2 \ \& \ (\forall z)(z \in H \to b_z = \sum_{k\in N}\sqrt{\lambda_k}\alpha_k(z,e_k)))\}.$$

□

The following corollary shows that T.Pitcher's one result (cf. [48],p.554) can be considered as a consequence of Corollary 4.10.

Corollary 4.11 *The measure μ is $B^{\frac{1}{2}}(H)$-quasiinvariant.*

Proof. We have

$$(\forall H_0)(\forall X)(\ H_0 \subseteq H \ \& \ \mathrm{card}(H_0) \le \aleph_0 \ \& \ X \in Ba(\mathbf{R}^{H_0}) \ \to$$

$$\to \mu(\{x : (\ell_z(x))_{z\in H_0} \in X\}) = \lambda(X \times \mathbf{R}^{H\setminus H_0})).$$

Set $H_0 = \{e_k : k \in N\}$. According to Corollary 4.10,

$$\lambda(X \times \mathbf{R}^{H\setminus H_0}) > 0$$

if and only if

$$\lambda(X \times \mathbf{R}^{H \setminus H_0} + (\sum_{k \in N} \sqrt{\lambda_k} \alpha_k (z, e_k))_{z \in H}) > 0$$

for arbitrary $(\alpha_k)_{k \in N} \in \ell_2$ and $X \in Ba(\mathbf{R}^{H_0})$.

Hence,

$$\mu(\{x : (\ell_{e_k}(x))_{k \in N} \in X\}) > 0$$

if and only if

$$\mu(\{x : (\ell_{e_k}(x))_{k \in N} \in X + (\sum_{k \in N} \sqrt{\lambda_k} \alpha_k)_{k \in N}\}) > 0$$

for arbitrary $(\alpha_k)_{k \in N} \in \ell_2$ and $X \in B(\mathbf{R}^{H_0})$.

Since the sets of the form $\{x : (\ell_{e_k}(x))_{k \in N} \in X\}$ $(X \in Ba(\mathbf{R}^{H_0}))$ coincides with the Baire σ-algebra of subsets of H, we conclude that the measure μ is $B^{\frac{1}{2}}(H)$-quasiinvariant. $\square$

Corollary 4.12 *The Wiener measure $\mu^W_{[0;2\pi]}$ on $\mathbf{R}^{[0;2\pi]}$ defined by Paley-Wiener (cf.*[64], *p.83, Chapter* 2.3*) is $A_0(\ell_2(J))$-quasiinvariant, where*

$$A_0(\ell_2(J)) = \{f \ : \ f \in \mathbf{R}^{[0;2\pi]} \ \& \ (\exists (x_j)_{j \in J} \in \ell_2(J) \ \&$$

$$\& \ (\forall t)(t \in [0;2\pi] \rightarrow f(t) = \sum_{j \in J} a_{tj} x_j))\},$$

$J = Z$ and A_0 is an operator generated by the matrix $\mathcal{A}_0 = (a_{tj})_{t \in [0;2\pi], j \in J}$ such that

$$(\forall t)(\forall j)(t \in [0;2\pi] \ \& \ j \in J \ \rightarrow a_{tj} = \begin{cases} \frac{t}{\sqrt{2\pi}}, & \textit{if } j = 0, \\ \frac{\sin(nt)}{\sqrt{2\pi}n}, & \textit{if } j = 2n, \\ \frac{\cos(nt)-1}{\sqrt{2\pi}n}, & \textit{if } j = 2n+1. \end{cases}).$$

Proof. Let us consider the Gaussian Baire measure $\mu^{G(A_0)}_{[0;2\pi]}$. Using Paley-Wiener construction, we establish that the Gaussian Baire measure $\mu^{G(A_0)}_{[0;2\pi]}$ is just the Wiener measure $\mu^W_{[0;2\pi]}$ defined on $\mathbf{R}^{[0;2\pi]}$. Applying Theorem 4.5, we conclude that the Wiener measure $\mu^W_{[0;2\pi]}$ is $A_0(\ell_2(J))$-quasiinvariant. $\square$

Remark 4.10 Following R.H.Cameron and W.T.Martin [19], the Wiener measure $\mathbf{meas}_W$ defined on the space C of all functions $x(t)$ continuous in $[0,1]$ and vanishing at $t = 0$, satisfies the condition

$$\mathbf{meas}_W(X + x_0) = e^{-\int_0^1 x_0'^2(t)dt} \int_X e^{-2\int_0^1 x_0'(t)dZ(t)} d \ \mathbf{meas}_W(Z),$$

where $X \in Ba(C[0,1]), x_0 \in C[0,1]$ is an absolutely continuous on $[0,1]$ and $x_0'(t)$ is equivalent to a function which is of bounded variation on $[0,1]$. This formula describes a class K of translations of $C[0,1]$ under which the Wiener measure is quasiinvariant. A natural embedding of C in $\mathbf{R}^{[0,1]}$ defines a Wiener Baire measure $\mu^W_{[0,1]}$ on $\mathbf{R}^{[0,1]}$. Note that the group of all admissible translations(in the sense of quasiinvariance) of $\mathbf{meas}_W$ and of the Baire

measure $\mu^W_{[0,1]}$ (notwithstanding, these measures have difference domains) coincide. Note also that the equality

$$(\forall X)(X \in Ba(\mathbf{R}^{[0,1]}) \to \mu^W_{[0,1]}(X) = \mu^W_{[0,2\pi]}(X \times \mathbf{R}^{]1,2\pi]}))$$

holds.

If $f \in A_0(\ell_2(J))$, then we have

$$\mu^W_{[0,1]}(X + (f(t))_{t\in[0,1]}) = \mu^W_{[0,2\pi]}((X \times \mathbf{R}^{]1,2\pi]} + f)).$$

Last relation means that $(f(t))_{t\in[0,1]}$ is an admissible translation(in the sense of quasi-invariance) of $\mu^W_{[0,1]}$. Hence, $\mu^W_{[0,1]}$ (correspondingly, $\mathbf{meas}_W$) is K_1-quasiinvariant, where K_1 is defined by

$$K_1 = \{(f(t))_{t\in[0,1]} : (f(t))_{t\in[0,2\pi]} \in A_0(\ell_2(J))\}.$$

Let us show that $K \subset K_1$ and $K_1 \setminus K \neq \emptyset$. Indeed, let $f \in K$. We set $\overline{f}(t) = f'(t)$, if $t \in [0,1]$, and $= f(1)$ if $t \in]1,2\pi]$. As the family $((a'_{jt})_{t\in[0,2\pi]})_{j\in J}$ is the basis in $L_2([0,2\pi])$ and $\overline{f}$ is equivalent to a function which is of bounded variation on $[0,2\pi]$, there exists $(x_j)_{j\in J} \in \ell_2(J)$ such that $\sum_{j\in J}(a'_{tj})_{t\in[0,2\pi]}x_j$ converges to $\overline{f}$ in the sense of $L_2([0,2\pi])$. Now we set $f^*(t) = \sum_{j\in J} x_j a_{tj}$ for $t \in [0,2\pi]$. As $f^* \in C$ and $f^{*'}(t) = f'(t)$ for $t \in [0,1]$, we deduce that $f^* = f$. Hence, $f \in K_1$. Now show that $K_1 \setminus K \neq \emptyset$. In this context let us consider a function $h \in C[0,1]$ defined by: $h(t) = \sin(\frac{1}{t})$, if $t \in]0,2\pi]$, and $= 0$ if $t = 0$. It is clear that $h \in L_2[0,2\pi]$ and h is no equivalent to a function which is of bounded variation on $[0,2\pi]$. Hence, there exists $(x_j)_{j\in J} \in \ell_2(J)$ such that $\sum_{j\in J} x_j(a'_{tj})_{t\in[0,2\pi]}$ converges to h in the sense of $L_2[0,2\pi]$. Set $h^*(t) = \sum_{j\in J} x_j a_{tj} = \int_0^t \sin(\frac{1}{\tau})d\tau$ for $t \in [0,2\pi]$. As $h^* \in A_0(\ell_2(J))$, we conclude that $(h^*(t))_{t\in[0,1]} \in K_1$, but $(h^*(t))_{t\in[0,1]} \notin K$, because $h^{*'}$ is not equivalent to a function which is of bounded variation.

Chapter 5

Invariant Borel Measures in $\mathbf{R}^N$

Let $\mathbf{R}^N$ be the topological vector space of all real-valued sequences equipped with the Tykhonoff topology. Let us denote by $B(\mathbf{R}^N)$ the σ-algebra of all Borel subsets in $\mathbf{R}^N$.

Let $(a_i)_{i \in N}$ and $(b_i)_{i \in N}$ be sequences of real numbers such that

$$(\forall i)(i \in N \rightarrow a_i < b_i).$$

We put

$$A_n = \mathbf{R}_0 \times \cdots \times \mathbf{R}_n \times (\prod_{i>n} \Delta_i) \ ,$$

for $n \in N$, where

$$(\forall i)(i \in N \rightarrow \mathbf{R}_i = \mathbf{R} \ \& \ \Delta_i = [a_i; b_i[).$$

We put also

$$\Delta = \prod_{i \in N} \Delta_i.$$

For an arbitrary natural number $i \in N$, consider the Lebesgue measure μ_i defined on the space $\mathbf{R}_i$ and satisfying the condition $\mu_i(\Delta_i) = 1$. Let us denote by λ_i the normed Lebesgue measure defined on the interval Δ_i.

For an arbitrary $n \in N$, let us denote by ν_n the measure defined by

$$\nu_n = \prod_{1 \le i \le n} \mu_i \times \prod_{i>n} \lambda_i,$$

and by $\overline{\nu}_n$ the Borel measure in the space $\mathbf{R}^N$ defined by

$$(\forall X)(X \in B(\mathbf{R}^N) \rightarrow \overline{\nu}_n(X) = \nu_n(X \cap A_n)).$$

The following assertion is valid.

Lemma 5.1 *For an arbitrary Borel set $X \subseteq \mathbf{R}^N$ there exists a limit*

$$\nu_\Delta(X) = \lim_{n \to \infty} \overline{\nu}_n(X).$$

Moreover, the functional ν_Δ is a nontrivial σ-finite measure defined on the Borel σ-algebra $B(\mathbf{R}^N)$.

Proof. First, observe that, for an arbitrary natural number n, the condition $A_n \subset A_{n+1}$ is valid. By the property of σ-additivity of the measure ν_{n+1}, we obtain

$$\overline{\nu}_{n+1}(X) = \nu_{n+1}(X \cap A_{n+1}) = \nu_{n+1}(X \cap [A_{n+1} \setminus A_n] \cup A_n) =$$

$$= \nu_{n+1}[X \cap (A_{n+1} \setminus A_n)] + \nu_{n+1}(X \cap A_n).$$

Note that the restriction $\nu_{n+1}|A_n$ of the measure ν_{n+1} to the set A_n coincides with the measure ν_n.

Indeed, we have

$$\nu_{n+1}(A_n \cap X) = (\prod_{1 \le i \le n+1} \mu_i \times \prod_{i > n+1} \lambda_i)(A_n \cap X) =$$

$$= \{\prod_{1 \le i \le n} \mu_i \times [\mu_{n+1}|\Delta_{n+1} + \mu_{n+1}|\{R \setminus \Delta_{n+1}\}] \times \prod_{i > n+1} \lambda_i\}(A_n \cap X) =$$

$$= (\prod_{1 \le i \le n} \mu_i \times \prod_{i > n} \lambda_i)(A_n \cap X) + (\prod_{1 \le i \le n} \mu_i \times (\mu_{n+1}|\{R \setminus \Delta_{n+1}\}) \times$$

$$\times \prod_{i > n+1} \lambda_i)(A_n \cap X) = \nu_n(A_n \cap X).$$

Since for an arbitrary $n \in N$ the inclusion $A_n \subset A_{n+1}$ holds, we have

$$(\forall X)(X \in B(\mathbf{R}^N) \to \nu_n(A_n \cap X) \le \nu_{n+1}(A_{n+1} \cap X)).$$

Hence there exists a limit $\lim_{n \to \infty} \overline{\nu}_n(X)$ which we denote by $\nu_\Delta(X)$.

Establish the following properties of ν_Δ.

I) The functional ν_Δ is countably additive.

Let $X = \cup_{k \in N} X_k$, where

$$(\forall m)(\forall p)(m \in N \,\&\, p \in N \,\&\, m \ne p \to X_m \cap X_p = \emptyset),$$

and, for an arbitrary $k \in N$, let X_k be a Borel subset of the space $\mathbf{R}^N$. Then we obtain

$$\nu_\Delta(\cup_{k \in N} X_k) = \sum_{k \in N} \nu_\Delta(X_k).$$

Indeed, on the one hand, we have

$$\nu_\Delta(X) = \lim_{n \to \infty} \overline{\nu}_n(X) = \lim_{n \to \infty} \overline{\nu}_n(\cup_{k \in N} X_k) = \lim_{n \to \infty} \sum_{k \in N} \overline{\nu}_n(X_k) \le$$

$$\le \sum_{k \in N} \lim_{n \to \infty} \overline{\nu}_n(X_k) = \sum_{k \in N} \nu_\Delta(X_k).$$

On the other hand, we have

$$\nu_\Delta(X) = \lim_{n \to \infty} \sum_{k \in N} \overline{\nu}_n(X_k) = \lim_{n \to \infty} (\sum_{k=1}^{m} \overline{\nu}_n(X_k)) +$$

$$+ \lim_{n \to \infty} \sum_{k > m} \overline{\nu}_n(X_k) =$$

$$= \sum_{k=1}^{m} \nu_\Delta(X_k) + \lim_{n\to\infty} \sum_{k>m} \overline{\nu}_n(X_k),$$

i.e.,

$$\nu_\Delta(\cup_{k\in N} X_k) \geq \sum_{1\leq k\leq m} \nu_\Delta(X_k).$$

Accordingly,

$$\nu_\Delta(\cup_{k\in N} X_k) \geq \sum_{k\in N} \nu_\Delta(X_k).$$

Thus, we have

$$\nu_\Delta(\cup_{k\in N} X_k) = \sum_{k\in N} \nu_\Delta(X_k).$$

II). The measure ν_Δ is nontrivial, since

$$\nu_\Delta(\Delta) = 1.$$

III). The measure ν_Δ is σ-finite. Indeed, we have

$$\mathbf{R}^N = (\mathbf{R}^N \setminus \cup_{n\in N} A_n) \cup (\cup_{n\in N} A_n).$$

Since $\mathbf{R}^N \setminus \cup_{n\in N} A_n \in B(\mathbf{R}^N)$, by the definition of the measure ν_Δ we have

$$\nu_\Delta(\mathbf{R}^N \setminus \cup_{k\in N} A_k) = \lim_{n\to\infty} \overline{\nu}_n((\mathbf{R}^N \setminus \cup_{k\in N} A_k) \cap A_n) = \lim_{n\to\infty} \nu_n(\emptyset) = 0.$$

Since, for an arbitrary natural number $n \in N$, the measure $\overline{\nu}_n$ is σ-finite, there exists a countable family $(B_k^{(n)})_{k\in N}$ of Borel measurable subsets of the space $\mathbf{R}^N$ such that

$$a)(\forall k)(k \in N \to \overline{\nu}_n(B_k^{(n)}) < +\infty);$$

$$b)(\forall n)(n \in N \to A_n = \cup_{k\in N} B_k^{(n)}).$$

Let us consider the family $(B_k^{(n)})_{k,n\in N}$.

It is clear that

$$(\forall k)(\forall n)(k \in N \;\&\; n \in N \to \nu_\Delta(B_k^{(n)}) = \overline{\nu}_n(B_k^{(n)}) < +\infty).$$

On the other hand, we have

$$\cup_{n\in N} A_n = \cup_{n\in N} \cup_{k\in N} B_k^{(n)},$$

i.e.,

$$\mathbf{R}^N = (\mathbf{R}^N \setminus \cup_{n\in N} A_n) \cup (\cup_{n\in N, k\in N} B_k^{(n)}).$$

The proof of Lemma 5.1 is completed. □

Remark 5.1 The measure ν_Δ described in Lemma 5.1 can be regarded as an inductive limit of the family of invariant measures $(\overline{\nu}_n)_{n\in N}$.

Recall that an element $h \in \mathbf{R}^N$ is called an admissible translation in the sense of invariance for the measure ν_Δ if

$$(\forall X)(X \in B(\mathbf{R}^N) \to \nu_\Delta(X+h) = \nu_\Delta(X)).$$

We define

$$G_\Delta = \{h : h \in \mathbf{R}^N \ \&\ h \text{ is an admissible translation for } \nu_\Delta\}.$$

It is easy to show that G_Δ is a vector subspace of $\mathbf{R}^N$.

Remark 5.2 Following Sudakov[170] there does not exist a σ-finite translation-invariant Borel measure on $\mathbf{R}^N$. Hence, an object of interest was the problem of the existence of such σ-finite Borel measures which have everywhere dense (in $\mathbf{R}^N$) groups of admissible translations in the sense of invariance (following Gelfand, such measures are called invariant measures). We must say that the first construction of such measure belongs to A.B. Kharazishvili (see [87]).

Our next theorem gives a representation of the algebraic structure of the vector subspace G_Δ of all admissible translations for Kharazishvili measure ν_Δ.

Theorem 5.1 *The following conditions are equivalent:*

$$1)\ \ g = (g_1, g_2, \cdots) \in G_\Delta,$$

$$2)\ \ \textit{the series} \sum_{i \in N} \frac{|g_i|}{b_i - a_i} \textit{ is convergent}.$$

Proof: Assume that for an element $g = (g_1, g_2, \cdots) \in \mathbf{R}^N$ the condition $1)$ is satisfied. Then we have

$$\nu_\Delta(\Delta + g) = \nu_\Delta(\Delta) = 1.$$

On the other hand, we have

$$\nu_\Delta(\Delta+g) = \nu_\Delta(\Delta+g) = \nu_\Delta(\prod_{i\in N}[a_i+g_i, b_i+g_i[) =$$

$$= \lim_{n\to\infty} \overline{\nu}_n(A_n \cap (\Delta+g)) = \lim_{n\to\infty}(\prod_{1\le i\le n}\mu_i \times \prod_{i>n}\lambda_i)((\prod_{1\le i\le n}\mathbf{R}_i \times$$

$$\times \prod_{i>n}[a_i, b_i[) \cap \prod_{i\in N}[a_i+g_i, b_i+g_i[) = \lim_{n\to\infty}(\prod_{1\le i\le n}\mu_i$$

$$(\prod_{1\le i\le n}[a_i+g_i, b_i+g_i[)) \times (\prod_{i>n}\lambda_i([a_i+g_i, b_i+g_i[)) =$$

$$= \lim_{n\to\infty}\prod_{i>n}\lambda_i([a_i, b_i[\cap[a_i+g_i, b_i+g_i[) = 1.$$

Let us show that

$$(\forall g)(g = (g_1, g_2, \cdots) \in G_\Delta \to \lim_{i\to\infty}\frac{|g_i|}{|b_i - a_i|} = 0).$$

Indeed, if we assume the contrary, then there exist a countable subset $(n_k)_{k\in N}$ of N and a positive real number $\varepsilon > 0$, such that

$$(\forall k)(k \in N \rightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \varepsilon).$$

Let us choose a number $m > 0$ such that $\varepsilon \cdot m > 1$. Since $g \in G_\Delta$, we have

$$m \cdot g = (m \cdot g_1, m \cdot g_2, \cdots) \in G_\Delta.$$

In view of the property of σ-additivity of the measure ν_Δ, we obtain

$$\nu_\Delta(\Delta) = \nu_\Delta(\Delta + m \cdot g) = 1.$$

But note that

$$(\Delta + m \cdot g) \cap (\cup_{n\in N} A_n) = \emptyset.$$

Indeed, assume the contrary and take

$$(x_i)_{i\in N} \in (\Delta + m \cdot g) \cap (\cup_{n\in N} A_n).$$

Then it is clear that, for the n_k-th coordinate, we have

$$(\exists k_0)(k_0 \in N \rightarrow (\forall k)(k \geq k_0 \rightarrow (a_{n_k} + m \cdot g_{n_k} \leq$$

$$\leq x_{n_k} \leq b_{n_k} + m \cdot g_{n_k}) \ \&\ (a_{n_k} \leq x_{n_k} \leq b_{n_k}))).$$

On the other hand, the validity of the condition

$$(\forall k)(k \in N \rightarrow \frac{|g_{n_k}|}{b_{n_k} - a_{n_k}} > \varepsilon),$$

implies the validity of the relation

$$(\forall k)(k \in N \rightarrow m \cdot |g_{n_k}| > b_{n_k} - a_{n_k}),$$

which shows us that the intervals $[a_{n_k}, b_{n_k}[$ and $[a_{n_k} + g_{n_k}, b_{n_k} + g_{n_k}[$ have an empty intersection. Hence the condition $\lim_{i\to\infty} \frac{|g_i|}{b_i - a_i} = 0$ holds.

From the validity of the condition $\lim_{i\to\infty} \frac{|g_i|}{b_i - a_i} = 0$, we conclude that there exists a natural number n_g such that

$$(\forall i)(i > n_g \rightarrow \frac{|g_i|}{b_i - a_i} < 1)$$

and

$$(\forall i)(i > n_g \rightarrow \lambda_i([a_i, b_i[\cap [a_i + g_i, b_i + g_i[) = \frac{b_i - a_i - |g_i|}{b_i - a_i} = 1 - \frac{|g_i|}{b_i - a_i}).$$

Keeping in mind that

$$\lim_{p\to\infty} \prod_{i \geq n_g + p} (1 - \frac{|g_i|}{b_i - a_i}) = 1$$

and considering the logarithms of both sides, we have

$$\lim_{p\to\infty}\sum_{i\geq n_g+p} ln(1-\frac{|g_i|}{b_i-a_i})=0.$$

This means that the series $\sum_{i\geq n_g} ln(1-\frac{|g_i|}{b_i-a_i})$ is convergent and the validity of the implication $1)\to 2)$ is proved.

Now let us prove $2)\to 1)$.

Let n_g be a natural number such that the series $\sum_{i\geq n_g} ln(1-\frac{|g_i|}{b_i-a_i})$ is convergent.

Let us consider an arbitrary element X having the form

$$X=B\times\prod_{i>n}\Delta_i,$$

where $B\in B(\mathbf{R}^N)\ \ (n\in N)$.

The sets of these forms generate the Borel σ-algebra $B(A_n)$ of the space A_n, and the condition $B(A_n)=B(\mathbf{R}^N)\cap A_n$ holds. To prove the implication $2)\to 1)$, it is sufficient to show the validity of the condition

$$\nu_\Delta(X+g)=\nu_\Delta\{[(B\times\prod_{n+1\leq i\leq n_g+n}\Delta_i)+(g_1,\cdots,g_{n_g})]\times$$

$$\times\prod_{i>n_g+n}[a_i+g_i,b_i+g_i]\}=\lim_{n\to\infty}\prod_{i=1}^{n_g+n}\mu_i(B\times\prod_{n+1\leq i\leq n_g+n}\Delta_i)\times$$

$$\times\prod_{i>n_g+n}\lambda_i([a_i+g_i,b_i+g_i]\cap[a_i,b_i])=\nu_\Delta(B\times\prod_{i>n}\Delta_i)\times$$

$$\times\lim_{n\to\infty}\prod_{i>n_g+n}(1-\frac{|g_i|}{b_i-a_i})=\nu_\Delta(B\times\prod_{i>n}\Delta_i)=\nu_\Delta(X).$$

We have used the well known result from mathematical analysis

$$(\text{the series }\sum_{i\geq n_g} ln(1-\frac{|g_i|}{b_i-a_i})\text{ is convergent})\Leftrightarrow$$

$$\Leftrightarrow\lim_{n\to\infty}\prod_{i\geq n_g+n}(1-\frac{|g_i|}{b_i-a_i})=ln1\ \ \Leftrightarrow\lim_{n\to\infty}\prod_{i>n_g+n}(1-\frac{|g_i|}{b_i-a_i})=1.$$

The proof of the theorem is complete. □

Remark 5.3 Let $\mathbf{R}^{(N)}$ be the space of all finite sequences, i.e.,

$$\mathbf{R}^{(N)}=\{(g_i)_{i\in N}|(g_i)_{i\in N}\in\mathbf{R}^N\ \&\ \text{card}\{i|g_i\neq 0\}<\aleph_0\}.$$

It is clear that, on the one hand, for an arbitrary compact infinite-dimensional parallelepiped $\Delta=\prod_{k\in N}[a_k,b_k]$, we have

$$\mathbf{R}^{(N)}\subset G_\Delta.$$

On the other hand, $G_\Delta \setminus \mathbf{R}^{(N)} \neq \emptyset$, since an element $(g_i)_{i\in N}$ defined by

$$(\forall i)(i \in N \to g_i = (1 - \exp\{-\frac{b_i - a_i}{2^i}\} \times (b_i - a_i)))$$

belongs to the difference $G_\Delta \setminus \mathbf{R}^{(N)}$.

It is easy to show that the vector space G_Δ is everywhere dense in $\mathbf{R}^N$ with respect to the Tykhonoff topology, since $\mathbf{R}^{(N)} \subset G_\Delta$.

Definition 5.1 Let G_Δ and G be some vector subspaces of $\mathbf{R}^N$ such that

$$G_\Delta \subset G \subset \mathbf{R}^N.$$

Let $(G_i)_{i\in I} = G/G_\Delta$ denote the factor group. We say that a family of elements $(g_i)_{i\in I}$ is a selector of a factor group G/G_Δ if $g_i \in G_i$ for each index $i \in I$.

We have the following statement.

Theorem 5.2 *Assume that G $(G_\Delta \subset G \subset \mathbf{R}^N)$ is some vector subspace of the space $\mathbf{R}^N$.A G-invariant σ-finite Borel measure taking a nonzero value on the element Δ exists if and only if the cardinality of the factor group G/G_Δ is countable.*

Proof. Let us prove the necessity. Assume that λ is a G-invariant σ-finite Borel measure taking a nonzero value on the element Δ. Assume also that the cardinality of the factor group G/G_Δ is uncountable. Let $(g_\xi)_{\xi<\omega_1}$ be any selector of this factor group. Consider the family

$$(g_\xi(\Delta))_{\xi<\omega_1}.$$

Let us show that

$$(\forall \xi_1)(\forall \xi_2)(0 < \xi_1 < \xi_2 < \omega_1 \to \lambda(g_{\xi_1}(\Delta) \cap g_{\xi_2}(\Delta)) = 0).$$

Assume the contrary and let the condition

$$\lambda(g_{\xi_1^*}(\Delta) \cap g_{\xi_2^*}(\Delta)) > 0$$

hold for some cardinal numbers $0 < \xi_1^* < \xi_2^* < \omega_1$.

It is clear that

$$(\lambda((\Delta + g^*_{\xi_1}) \cap (\Delta + g_{\xi_2^*})) > 0) \to (\lambda(\Delta \cap (\Delta + g_{\xi_2^*} - g_{\xi_1^*})) > 0).$$

Applying the scheme proposed in Theorem 8.3, we can conclude that, for some positive real number q, the following equality is valid:

$$\lambda|_{\cup_{n\in N} A_n} = q \times \nu_\Delta.$$

Accordingly, we have $g_{\xi_2^*} - g_{\xi_1^*} \in G_\Delta$, which is a contradiction since the elements $g_{\xi_1^*}$ and $g_{\xi_2^*}$ belong to different classes of the factor group G/G_Δ.

Thus

$$(\forall \xi_1)(\forall \xi_2)(0 < \xi_1 < \xi_2 < \omega_1 \to \lambda(g_{\xi_1}(\Delta) \cap g_{\xi_2}(\Delta)) = 0).$$

Now it is easy to construct an ω_1-sequence of disjoint Borel subsets $(K_\xi)_{\xi<\omega_1}$ with a positive λ-measure and thus, we have a contradiction with the condition of the σ-finiteness of the measure λ.

The necessity is proved.

Let us now prove the sufficiency. Let $(g_k)_{k\in N}$ be a selector of the factor group G/G_Δ. Let us define the family of pairwise orthogonal G_Δ-invariant σ-finite measures by

$$(\forall X)(\forall k)(X \in B(\mathbf{R}^N)\ \&\ k \in N \to \mu_k(X) = \nu_\Delta(X - g_k)).$$

We can construct the measure λ by the formula

$$(\forall X)(X \in B(\mathbf{R}^N) \to \lambda(X) = \sum_{k\in N} \mu_k(X)).$$

It is easy to prove that the functional λ is a nontrivial G-invariant σ-finite Borel measure taking a nonzero value on the set Δ.

The proof of Theorem 5.2 is completed. $\square$

If we consider the classical Lebesgue measure ℓ_n in the finite-dimensional Euclidean space $\mathbf{R}^n$, then we easily conclude that the group $\mathbf{R}^n$ of all admissible translations of the measure ℓ_n is the ℓ_n- massive set being at the same time of second category.

The following theorem is of some interest.

Theorem 5.3 *Let G be a subgroup of $\mathbf{R}^N$ which cannot be covered by the union of a countable family of compact sets. Then any σ-finite G-invariant Borel measure is trivial.*

The proof of Theorem 5.3 can be found in [86].

Remark 5.4 Theorem 5.3 shows that the group of all admissible translations for an arbitrary σ-finite Borel measure defined on the topological vector space $\mathbf{R}^N$ is small in the sense of category, because an arbitrary compact subset of the space $\mathbf{R}^N$ is nowhere dense in this space and a countable union of such sets is of the first category.

The next theorem provides information (in measure-theoretical terms) about the group of all admissible translations for an arbitrary Borel measure μ defined on the measurable space $(\mathbf{R}^N, B(\mathbf{R}^N))$ and being invariant with respect to everywhere dense subspace of this space.

Theorem 5.4 *If the group G of all admissible translations of some nontrivial σ-finite Borel measure μ defined on the measurable space $(\mathbf{R}^N, B(\mathbf{R}^N))$ is everywhere dense in this space, then $\mu^*(G) = 0$.*

Proof. Assume $\mu^*(G) > 0$. Let us define the functional λ by

$$(\forall B)(B \in B(\mathbf{R}^N) \to \lambda(B) = \mu^*(A \cap G)).$$

Note that the projection λ_1 of the measure λ on the group G is the G-invariant σ-finite Borel measure defined on the measurable space $(G, B(G)) = (G, G \cap B(\mathbf{R}^N))$. By using one result of Xia Dao-Xing (see [173], p.70), we conclude that the group G is a locally compact topological vector space. This means that the dimension of the vector space G is finite and we obtain a contradiction to the condition of everywhere density of the group G in $\mathbf{R}^N$. $\square$

Remark 5.5 Theorem 5.4 is a partial case of the general result of Veršik [174] stating that, for an arbitrary probability Borel measure μ, defined on the infinite-dimensional topological vector space, the group Q_μ of all admissible (in the sense of quasiinvariance) translations for the measure μ is of μ^*-measure zero.

Now we consider the following question:

Does there exist a measure μ_Δ in $\mathbf{R}^N$ which would be equivalent to the canonical Gaussian measure defined in $\mathbf{R}^N$?

In this direction we have the following result.

Theorem 5.5 *For an arbitrary infinite-dimensional parallelepiped Δ in $\mathbf{R}^N$ the equality*

$$G_\Delta \neq \ell_2$$

holds.

Proof. Let $\Delta = \prod_{k=1}^{\infty} [a_k, b_k]$ be an arbitrary infinite-dimensional parallelepiped in $\mathbf{R}^N$ such that

$$k \in N \to b_k - a_k > 0.$$

It is possible to have only two cases:

I) The sequence $(b_k - a_k)_{k\in N}$ is bounded by a positive number m. Then the sequence

$$\left(\frac{b_k - a_k}{m \cdot k}\right)_{k\in N} \in \ell_2,$$

but

$$\sum_{k=n}^{\infty} ln\left(1 - \left|\frac{b_k - a_k}{m \cdot k}\right| \cdot \frac{1}{b_k - a_k}\right)$$

is not convergent for an arbitrary natural number n.

II) The sequence $(b_k - a_k)_{k\in N}$ is not bounded; then there exists an increasing subsequence $(n_k)_{k\in N}$ of natural numbers, such that

$$(\forall k)\left(k \in N \to \frac{(b_{n_k} - a_{n_k})^2}{k^4} > 1\right).$$

Let us define the sequence $(x_k)_{k\in N}$ by the formula

$$x_m = \begin{cases} \frac{b_{n_k} - a_{n_k}}{k^2}, & \text{if } m = n_k, \\ 0, & \text{if } m \neq n_k. \end{cases}$$

Then, on the one hand, it is clear that

$$\sum_{m\in N} x_m^2 = +\infty,$$

which implies

$$(x_m)_{m\in N} \notin \ell_2,$$

and, on the other hand, we have

$$\sum_{m>1}^{\infty} ln(1-\frac{|x_m|}{b_m-a_m}) = \sum_{k>1} ln(1-\frac{x_{n_k}}{b_{n_k}-a_{n_k}}) = \sum_{k>1} ln(1-\frac{1}{k^2}),$$

which, using Theorem 5.1, implies that $(x_m)_{m\in N} \in G_\Delta$. $\square$

Remark 5.6 Note that in Solovay's model no nontrivial translation-invariant Borel measure μ on $\mathbf{R}^N$ is equivalent to the canonical Gaussian Borel measure (see Corollary 4.4).

In context of Theorem 5.5 we posed the following:

Problem 5.1 *Does there exist a nontrivial translation-invariant Borel measure μ on $\mathbf{R}^N$ such that the restriction of μ to some Borel subset would be equivalent to the canonical Gaussian measure in $\mathbf{R}^N$?*

The problem of equivalence and orthogonality relations between two measures in infinite-dimensional topological vector spaces has been investigated by many authors. In this direction especially the result of S. Kakutani can be mentioned [79](see also Theorem 4.3) stated that if one has equivalent probability measure μ_i and ν_i on the σ-algebra $\mathcal{L}_i$ of subsets of a set $\Omega_i, i = 1,2,\cdots$ and if μ and ν denote respectively the infinite product measures $\prod_{i\in N}\mu_i$ and $\prod_{i\in N}\nu_i$ on the infinite product σ-algebra generated on the infinite product set Ω, then μ and ν are either equivalent or orthogonal. Similar dichotomies have revealed themselves in the study of Gaussian stochastic processes. C. Cameron and W.E. Martin proved in [19] that if one considers the measures induced on a path space by a Wiener process on the unit interval, then if the variances of the processes are different the measures are orthogonal. These sort of results were generalized by many authors(cf.[19],[20],[39],[52] and others). A.M. Vershik [174] proved that a group of all admissible translations(in the sense of quasiinvariance) of arbitrary Gaussian measure in infinite-dimensional separable Hilbert space is a linear manifold. In this context I.I. Gikhman and A.V. Skorokhod has been considered in [49,chapter 7,paragraph 2] the following problem:

Does there exist a probability Borel measure μ in ℓ_2 which satisfies the following conditions:

(i) the group Q_μ of all admissible translations (in the sense of quasiinvariance) is an everywhere dense linear manifold in ℓ_2;

(ii) there exists $a \in \ell_2 \setminus Q_\mu$ such that a measure μ is not orthogonal to the measure $\mu^{(a)}$, where

$$(\forall X)(X \in B(\ell_2) \to \mu^{(a)}(X) = \mu(X-a))?$$

Gikhman-Skorokhod's solution of this problem was supported using the technique of Gaussian measures in infinite-dimensional separable Hilbert space.

In the next theorem we shall demonstrate how Gikhman-Skorokhod's above-mentioned result can be extended for invariant Borel measures in ℓ_2.

Theorem 5.6 *There exists a nonzero σ-finite Borel measure μ in ℓ_2 which satisfies the following conditions:*

(i) The group I_μ of all admissible (in the sense of invariance) translations for the measure μ is an everywhere dense linear manifold in ℓ_2.

(ii) There exists $a \in \ell_2 \setminus I_\mu$ such that a measure $\mu^{(a)}$ is not orthogonal to the measure μ, where

$$(\forall X)(X \in B(\ell_2) \rightarrow \mu^{(a)}(X) = \mu(X - a)).$$

Proof. According to Suslin's theorem we have $B(\ell_2) \subseteq B(\mathbf{R}^N)$. Set

$$(\forall X)(X \in B(\ell_2) \rightarrow \mu_1(X) = \nu_{\Delta_1}(X) \ \&\ \mu_2(X) = \nu_{\Delta_2}(X)),$$

where $\Delta_1 = \prod_{i\in N}[0, \frac{1}{i+1}[$ and $\Delta_2 = \prod_{i\in N}[0, \frac{1}{(i+1)2^i}[$. According to Theorem 5.1, we have

$$I_{\mu_1} = \{(g_i)_{i\in N} : (g_i)_{i\in N} \in \ell_2 \ \&\ \sum_{i\in N} |g_i|(i+1) < \infty\},$$

and

$$I_{\mu_2} = \{(h_i)_{i\in N} : (h_i)_{i\in N} \in \ell_2 \ \&\ \sum_{i\in N} |h_i|2^i(i+1) < \infty\},$$

where I_{μ_1} and I_{μ_2} denote groups of all admissible translations (in the sense of invariance) of μ_1 and μ_2, respectively. It is clear that $I_{\mu_2} \subset I_{\mu_1}$.

Let us show that measures μ_1 and μ_2 are orthogonal. Indeed, the measure μ_2 is concentrated on the set

$$F_2 = \cup_{n\in N}\mathbf{R}^N \times \prod_{i>n}[0, \frac{1}{(i+1)2^i}[$$

and $\mu_1(F_2) = 0$.

We set $\mu = \mu_1 + \mu_2$. Note that a group of all admissible translations(in the sense of invariance) I_μ of the measure μ coincides with I_{μ_2}. It is clear also that I_μ is the linear manifold. The fact that I_μ is everywhere dense in ℓ_2 is a simple consequence of the statement that the linear manifold of all finite sequences is everywhere dense in ℓ_2. If we consider an element $a = (\frac{1}{(1+i)2^i})_{i\in N} \in I_{\mu_1} \setminus I_{\mu_2}$, then a measure μ is not orthogonal to the measure $\mu^{(a)}$, because it has the absolutely continuous component $\mu_2^{(a)}$ with respect to the measure μ. $\square$

Remark 5.7 If we consider probability measures λ_1 and λ_2 which are equivalent to measures μ_1 and μ_2, respectively, then the measure $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$ would be a solution of the above-mentioned problem. Hence, Theorem 5.6 may be regarded as a generalization of above-mentioned result obtained in [48].

The following definition is important for the theory of invariant measures and its various applications.

Definition 5.2 Let (E, S, μ) be a measurable space with measure μ. Let G be a group of transformations of the space E such that

$$(\forall h)(\forall B)(h \in G \ \&\ B \in S \Rightarrow h(B) \in S).$$

We say that a set $B_0 \in S$ with $\mu(B_0) > 0$ satisfies the zero-one law with respect to the pair (μ, G) if

$$(\forall g)(g \in G \Rightarrow \frac{\mu(g(B_0))}{\mu(B_0)} = 1 \vee \frac{\mu(g(B_0))}{\mu(B_0)} = 0).$$

The zero-one law is realized in the following situation with the infinite-dimensional Hilbert space ℓ_2.

Theorem 5.7 *Let*

$$\triangle_0 = \prod_{i\in N}\Big[0;\frac{1}{2^{i+1}}\Big].$$

Then an arbitrary element $B \in \mathcal{B}(l_2)\cap(\cup_{n\in N}A_n)$ *with* $\mu_{\Delta_0}(B)>0$, *where*

$$(\forall n)(n\in N \Rightarrow A_n = \mathbf{R}^n \times \prod_{i\geq n+1}\Big[0;\frac{1}{2^{i+1}}\Big]),$$

satisfies the zero-one law with respect to the pair $(\nu_\triangle, l_2)$.

Proof. Note that it is sufficient to show the validity of the relation

$$(\forall B)(\forall h)(B\in\mathcal{B}(l_2)\cap(\bigcup_{n\in N}A_n)\ \&\ \nu_{\triangle_0}(B)>0\ \&$$

$$h\in l_2\setminus G_{\triangle_0}\Rightarrow \mu(B+h)=0).$$

Since $h=(h_1,h_2,\cdots)\in l_2\setminus G_{\triangle_0}$, for an arbitrary natural number $k\in N$ the series

$$\sum_{n=k}^{\infty}\ln(1-|h_n|\times 2^{n+1})$$

is not defined or for some natural number k_0 the series $\sum_{n=k_0}^{\infty}\ln(1-|h_n|\times 2^{n+1})$ is divergent.

In the first case, there exists a subsequence $(n_p)_{p\in N}$ of N such that

$$1-|h_{n_p}|\times 2^{n_p+1}<0,$$

i.e., $|h_{n_p}|>\frac{1}{2^{n_p+1}}$. The second case means that

$$(\prod_{i\in N}\Big[0;\frac{1}{2^{i+1}}\Big]+h)\cap(\bigcup_{n\in N}A_n)=\emptyset,$$

from which we have

$$(\bigcup_{n\in N}A_n+h)\cap(\bigcup_{n\in N}A_n)=\emptyset.$$

In particular,

$$(B+h)\cap\bigcup_{n\in N}A_n=\emptyset.$$

Consider the second case where, for some $k_0\in N$, the series

$$\sum_{n\geq k_0}\ln(1-|h_n|\times 2^{n+1})$$

is divergent.

Let us consider the following possible cases.

1). We can choose a subsequence $(n_p)_{p\geq 1}$ of natural numbers such that:

$a)$ $\sum_{p=1}^{\infty} \ln(1-|h_{n_p}| \times 2^{n_p+1}) = -\infty$,
$b)$ $0 < 1-2^{n_p+1} \times |h_{n_p}| < 1$.

2). There exists a subsequence $(n_m)_{m\geq 1}$ of natural numbers such that:
$c)$ $\sum_{m=1}^{\infty} \ln(1-|h_{n_m}| \times 2^{n_m+1}) = +\infty$,
$d)$ $1 < 1-|h_{n_m}| \times 2^{n_m+1}$.

Note that the case $2)$ is not possible since

$$|h_{n_m}| \times 2^{n_m+1} \geq 0.$$

In the case $1)$ there exists a subsequence $(n_{p_l})_{l\in N}$ of $(n_p)_{p\in N}$ such that

$$\left|h_{n_{p_{l+1}}}\right| \leq 2 \times \left|h_{n_{p_l}}\right|.$$

Hence we have

$$(\bigcup_{n\in N} A_n + h) \cap \bigcup_{n\in N} A_n = \emptyset.$$

The last relation implies the validity of the condition

$$(\forall B)(B \in \mathcal{B}(l_2) \cap \bigcup_{n\in N} A_n \Rightarrow \nu_{\triangle_0}((B+h) \cap \bigcup_{n\in N} A_n) = 0).$$

Since

$$\nu_{\triangle_0}(B+h) = \nu_{\triangle_0}((B+h) \setminus \bigcup_{n\in N} A_n) + \nu_{\triangle_0}((B+h) \cap (\bigcup_{n\in N} A_n)),$$

we obtain

$$\nu_{\triangle_0}(B+h) = 0,$$

which completes the proof of Theorem 5.7. $\square$

Remark 5.8 Note that not all elements $B \in \mathcal{B}(l_2)$ satisfy the zero-one law with respect to the pair $(\nu_{\triangle_0}, l_2)$. Indeed, if we consider the element $B_0 \in \mathcal{B}(l_2)$ defined by

$$B_0 = \triangle_0 \cup (\triangle_0 + (2,1,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots)) \cup$$

$$\cup (\triangle_0 + (1,2,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots)),$$

then, on the one hand,

$$\nu_{\triangle_0}(B_0) = 1$$

and, on the other hand, for $h = (1,1,1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots)$, we have

$$\frac{\nu_{\triangle_0}(B_0+h)}{\nu_{\triangle_0}(B_0)} = 2,$$

which means that the set B_0 does not satisfy the zero-one law with respect to the pair $(\nu_{\triangle_0}, l_2)$.

It will be interesting to describe the class of all Borel subsets of the space ℓ_2 which satisfy the zero-one law with respect to the pair $(\nu_{\triangle_0}, l_2)$.

Remark 5.9 In context of Theorem 5.7 see Theorem 10.1.

It is well-known that, in the Euclidean space $\mathbf{R}^n$, nonmeasurable (in the sense of the Lebesgue measure) subsets preserve some properties under translations. Among of them can be mentioned Baire's property of non-measurability in the sense of Lebesgue, Vitali's property, Bernstein's property and so on. Here we discuss analogous questions in the infinite dimensional topological vector space $\mathbf{R}^N$ for above-mentioned invariant measures.

The following notions are important for our further investigation of certain sets in the infinite-dimensional topological vector space $\mathbf{R}^N$.

Definition 5.3 We say that a set $X \subseteq \mathbf{R}^N$ has the Baire property if there exist sets $G \subseteq \mathbf{R}^N$, $X_1 \subseteq \mathbf{R}^N$ and $X_2 \subseteq \mathbf{R}^N$ such that

$$X = (G \setminus X_1) \cup X_2,$$

where G is an open set in $\mathbf{R}^N$, X_1 and X_2 are some subsets of $\mathbf{R}^N$ of first category.

Definition 5.4 A set $Y \subseteq \mathbf{R}^N$ is called a Bernstein set if $(\forall F)(F \subseteq \mathbf{R}^N \;\&\; F$ is closed in $\mathbf{R}^N \;\&\; Card(F) = c \Rightarrow F \cap Y \neq \emptyset \;\&\; F \cap (\mathbf{R}^N \setminus Y) \neq \emptyset)$.

Definition 5.5 A set $Z \subseteq \mathbf{R}^N$ is called a Vitali set if there exist a vector subspace $G \subseteq \mathbf{R}^N$ and a G-invariant σ-finite non-trivial Borel measure μ such that for some μ-measurable set X with $\mu(X) > 0$ and for some countable subgroup $G_0 \subseteq G$ the following conditions are satisfied:

a) a set Y intersects every set of the factor space G/G_0 at most in one point;

b) $Y \subseteq X \;\&\; X \subseteq \bigcup_{g \in G_0} g(Y)$.

The following theorem asserts that the subsets in $\mathbf{R}^N$ preserve Baire's and Bernstein's properties under translations.

Theorem 5.8 *The following conditions hold:*

a) $(\forall X)(\forall h)(X \subseteq \mathbf{R}^N \;\&\; h \in \mathbf{R}^N \;\&\; (X$ *has the property of Baire*$) \Rightarrow (X + h$ *has the property of Baire*$))$;

b) $(\forall X)(\forall h)(X \subseteq \mathbf{R}^N \;\&\; h \in \mathbf{R}^N \;\&\; (X$ *is a Bernstein set*$) \rightarrow (X + h$ *is a Bernstein set*$))$.

The proof of Theorem 5.8 is quite an easy one.

Definition 5.6 Let (E, G, S) be an invariant measurable space. We recall that a group G of transformations acts freely in E if

$$(\forall x)(\forall g)(x \in E \;\&\; g \in G \;\&\; g \neq Id_E \;\rightarrow g(x) \neq x),$$

where Id_E denotes an identity transformation of E.

Theorem 5.9 *Let μ be an arbitrary nontrivial G-invariant $(G \subseteq \mathbf{R}^N)$ Borel measure and let a group G contains a freely acting uncountable subgroup of the additive group $\mathbf{R}^N$. If we denote by L the completion of the class $\mathcal{B}(\mathbf{R}^N)$ by the measure μ, then* $(\exists h_0)(\exists Y)(h_0 \in \mathbf{R}^N \;\&\; Y$ *is a Vitali set* $\;\&\; Y \in L \rightarrow Y + h_0 \notin L)$.

Proof. Let us consider a probability measure μ_0 which is equivalent to the measure μ. Using Theorem 5.3, we have

$$(\exists B_0)(\exists h_0)(B_0 \in \mathcal{B}(\mathbf{R}^N) \ \&\ \mu_0(B) > 0 \ \&\ h_0 \in \mathbf{R}^N \Rightarrow \mu_0(B_0 + h_0) = 0).$$

Since the measure μ_0 is equivalent to the measure μ, we get

$$\mu(B_0) > 0 \quad \& \quad \mu(B_0 + h_0) = 0.$$

It is clear that, for the completion $\overline{\mu}$ of the measure μ, we have

$$\overline{\mu}(B_0) > 0 \quad \& \quad \overline{\mu}(B_0 + h_0) = 0.$$

By Lemma 8.1, we conclude that there exists a μ-nonmeasurable Vitali set $Y \subseteq B_0$. On the other hand, since $Y_0 + h_0 \subseteq B_0 + h_0$ and $\overline{\mu}$ is completion of μ, we obtain

$$Y + h_0 \in L \quad \& \quad Y \notin L. \quad \square$$

Remark 5.10 Theorem 5.9 states that the σ-algebra of subsets of the space $\mathbf{R}^N$ obtained by the operation of completion (with respect to a certain σ-finite invariant Borel measure) is not an $\mathbf{R}^N$-invariant σ-algebra. An analogue of this proposition is not true in the finite-dimensional Euclidean space $\mathbf{R}^n$ (It is sufficient to consider the Lebesgue measure l_n $(n \geq 1)$).

Remark 5.11 An analogue of Theorem 5.8 is not valid in the Euclidean space $\mathbf{R}^n$. Analogous result is not true also for Bernstein subsets in $\mathbf{R}^n$.

Summarizing the results of theorems 5.8 and 5.9, we conclude that the nonmeasurability property of Vitali subsets is not preserved under translations, but the property of absolute nonmeasurability of the Bernstein set is preserved in that case.

The following statement shows us that the nonmeasurability of Vitali sets is not preserved under transformations from a given group.

Theorem 5.10 *Assume that (E, G, S, μ) is a space with a non-zero σ-finite G-invariant measure. Let the group G contains an uncountable freely acting subgroup and G_0 be a group of transformations of the space E with respect to which the measure μ is not G_0-quasiinvariant. Then there exists a Vitali set Y such that:*
a) $Y \notin dom(\overline{\mu})$;
b) $(\exists h_0)(h_0 \in G_0 \to h_0 + Y \in dom(\overline{\mu}))$.

The proof of Theorem 5.10 is based also on Lemma 8.1.

The following lemma plays a key role in our further investigations.

Lemma 5.2 *Let μ be an arbitrary non-zero σ-finite continuous Borel measure defined on the uncountable Polish space. If we denote by $\overline{\mu}$ the completion of the measure μ, then the following relation is valid:*

$$(\forall X)(\forall Y)(X \textit{ is a Bernstein set} \ \&\ \overline{\mu}(Y) > 0 \to X \cap Y \notin dom(\overline{\mu})).$$

The proof of Lemma 5.2 can be obtained as follows: if $X \cap Y \in \mathrm{dom}(\overline{\mu})$, then, applying inner regularity of the measure μ and the equality $\mu(Y) = \mu^*(Y \cap X)$, we conclude that there

exists a compact subset $F \subseteq X \cap Y$ with $\mu(F) > 0$. Since μ is continuous, we establish that card(F) $= c$. But it is not possible because the set $\mathbf{R}^N \setminus X$ (which is also a Bernstein set) does not intersect F.

The main property of Bernstein sets in infinite-dimensional vector space $\mathbf{R}^N$ is presented in the next theorem.

Theorem 5.11 *Let μ be a non-zero σ-finite continuous Borel measure defined on the space $\mathbf{R}^N$. Assume X to be an arbitrary Bernstein set. Then there exists a subset $X_0 \subseteq X$ such that*

$$a)\ \ X_0 \notin dom(\overline{\mu}),$$

$$b)\ \ (\exists h_0)(h_0 \in \mathbf{R}^N \Rightarrow X_0 + h_0 \in dom(\overline{\mu})),$$

where $\overline{\mu}$ denotes the completion of the measure μ.

Proof. Without loss of generality, we may assume that μ is a probability Borel measure. According to Ulam's well-known result (cf. Lemma 10.1), the measure μ is concentrated on a countable union $\bigcup_{k\in N} K_n$ of compact subsets of the space $\mathbf{R}^N$. Following Lemma 10.3, there exists a translation $h_0 \in \mathbf{R}^N$ such that

$$(\bigcup_{n\in N} K_n) \cap (\bigcup_{n\in N} K_n + h_0) = \emptyset.$$

Let X be an arbitrary Bernstein subset of the space $\mathbf{R}^N$. Consider $X_0 = X \cap \bigcup_{n\in N} K_n$. By Lemma 5.2, we have

$$X_0 \notin \mathrm{dom}(\overline{\mu}).$$

On the other hand, since $X_0 + h_0 \subseteq \mathbf{R}^N \setminus \cup_{n\in N} K_n$, by the property of the measure $\overline{\mu}$, we obtain

$$X_0 + h_0 \in \mathrm{dom}(\overline{\mu}). \quad \square$$

Remark 5.12 Theorems 5.10 and 5.11 show us that the domain of the completion of an arbitrary non-zero σ-finite continuous Borel measure μ is not $\mathbf{R}^N$-invariant.We can observe also that an arbitrary non-zero continuous Borel measure on $\mathbf{R}^N$is not complete.

Definition 5.7 Let E be a base space, G be a group of transformations of E and let X be a subset of the space E. X is called a G-absolutely negligible set if for any G-invariant σ-finite measure μ, there exists its $G-$invariant extension $\overline{\mu}$ such that $X \in \mathrm{dom}(\overline{\mu})$ and $\overline{\mu}(X) = 0$.

A geometrical characterization of absolutely negligible subsets, due to A.B. Kharazishvili, is presented in the next theorem.

Theorem 5.12 *Let E be a base space, G be a group of transformations of E containing some uncountable subgroup acting freely in E, and X be an arbitrary subset of the space E. Then the following two conditions are equivalent:*

1) X is a G-absolutely negligible subset of the space E;

2) for an arbitrary countable G-configuration X' of the set X, there exists a countable sequence $(g_k)_{k\in N}$ of elements of G such that

$$\bigcap_{k\in N} g_k(X') = \emptyset.$$

The proof of Theorem 5.12 can be found in [85].

We have the following theorem.

Theorem 5.13 *Let E_1 be a nonempty base space, G_1 be a group of transformations of E_1, containing an uncountable subgroup acting freely in E_1, and X_1 be a G_1-absolutely negligible subset of the space E_1. Further, let E_2 be an arbitrary nonempty set and G_2 be an arbitrary group of transformations of E_2. Then for any subset $X_2 \subseteq E_2$ the set*

$$X_1 \times X_2$$

is a $G_1 \times G_2$-absolutely negligible subset of the space $E_1 \times E_2$.

Proof. Note that a group $G_1 \times G_2$ satisfies all conditions in Theorem 5.12. Let $(\widetilde{g}_k)_{k\in N} = (g_k^{(1)}, g_k^{(2)})_{k\in N}$ be an arbitrary countable sequence of elements of the group $G_1 \times G_2$. Consider

$$X' = \bigcup_{k\in N} g_k^{(1)}(X_1).$$

By Theorem 5.12, we conclude that there exists a countable subgroup $(h_p^{(1)})_{p\in N}$ such that

$$\bigcap_{p\in N} h_p^{(1)}(\bigcup_{k\in N} g_k^{(1)}(X_1)) = \emptyset.$$

Let h be an arbitrary element of the group G_2. Then it is easy to prove that

$$\bigcup_{k\in N}(g_k^{(1)}, g_k^{(2)})(X_1 \times X_2) \subseteq \bigcup_{k\in N}(g_k^{(1)}, g_k^{(2)})(X_1 \times X_2) \subseteq$$

$$\subseteq \bigcup_{k\in N} g_k^{(1)}(X_1) \times E_2.$$

On the other hand, we have

$$\bigcap_{p\in N}(h_p^{(1)}, h)(\bigcup_{k\in N}(g_k^{(1)}, g_k^{(2)})(X_1 \times X_2)) \subseteq$$

$$\subseteq \bigcap_{p\in N}(h_p^{(1)}(\bigcup_{k\in N} g_k^{(1)}(X_1))) \times E_2.$$

Using the condition

$$\bigcap_{p\in N} h_p^{(1)}(\bigcup_{k\in N} g_k^{(1)}(X_1)) = \emptyset,$$

we obtain

$$\bigcap_{p\in N}(h_p^{(1)}, h)(\bigcup_{k\in N}(g_k^{(1)}, g_k^{(2)})(X_1 \times X_2)) \subseteq \emptyset.$$

By Theorem 5.12, we conclude that the set $X_1 \times X_2$ is a $G_1 \times G_2$-absolutely negligible subset of the space $E_1 \times E_2$, and Theorem 5.13 is proved. □

The following result also belongs to A.B. Kharazishvili(cf.[85]).

Theorem 5.14 *Let Γ be an arbitrary uncountable subgroup of the additive group R. Then there exists a countable family $(X_n)_{n\in N}$ of subsets of R such that:*
1) $(\forall n)(n \in N \Rightarrow ($ *the set X_n is a Γ-absolutely negligible subset of R*$)$;
2) $\mathbf{R} = \bigcup_{n\in N} X_n$.

We have the following proposition.

Theorem 5.15 *Let $\mathbf{R}^N$ be the space of all sequences of real numbers, $\mathbf{R}^{(N)}$ be the group of all finite sequences of the space $\mathbf{R}^N$, i.e.,*

$$\mathbf{R}^{(N)} = \{(x_k)_{k\in N} | (x_k)_{k\in N} \in \mathbf{R}^N \ \& \ Card\{k : x_k \neq 0\} < \aleph_0\}.$$

Then there exists a countable family $(Z_n)_{n\geq 1}$ of subsets of $\mathbf{R}^N$ such that:
1) $(\forall n)(n \in N \Rightarrow ($ *the set Z_n is $R^{(N)}$-absolutely negligible subset of $\mathbf{R}^N$*$))$.
2) $\mathbf{R}^N = \bigcup_{n\in N} Z_n$.

The proof of Theorem 5.15 will be obviously obtained if we use theorems 5.13,5.14 and the equality

$$\mathbf{R}^{(N)} = \mathbf{R} \times \mathbf{R}^{(N\setminus\{0\})}.$$

Let E be a base space and let Γ be a group of its transformations.

A set $Y \subseteq E$ is said to be Γ-absolutely nonmeasurable if, for every nontrivial σ-finite Γ-invariant (Γ-quasiinvariant) measure μ defined on E, we have $Y \notin \mathrm{dom}(\mu)$.

One simple method of the construction of absolutely nonmeasurable subsets in product spaces is presented in the following theorem.

Theorem 5.16 *Let E_1 be a base space, Γ_1 be a group of transformations of E_1, Y_1 be a Γ_1-absolutely nonmeasurable subset of the space E_1. Further, let E_2 be also a base space and Γ_2 be a group of its transformations. Then a subset $Y_1 \times E_2$ of the space $E_1 \times E_2$ is $\Gamma_1 \times \Gamma_2$-absolutely nonmeasurable.*

Proof. Let μ be a $\Gamma_1 \times \Gamma_2$-quasiinvariant probability measure defined on $E_1 \times E_2$ such that

$$Y_1 \times E_2 \in \mathrm{dom}(\mu).$$

It is clear that the class $\mathcal{F}$ defined by the formula

$$\mathcal{F} = \{A : A \in \mathrm{dom}(\mu) \ \& \ A = X \times E_2\}.$$

is a $\Gamma_1 \times \Gamma_2$-invariant σ-algebra of subsets of the space $E_1 \times E_2$ and $\mathcal{F} \subseteq dom(\mu)$.

Let us consider the class $\mathcal{F}_1$ defined by

$$\mathcal{F}_1 = \{X : X \subseteq E_1 \ \& \ X \times E_2 \in \mathcal{F}\}.$$

It is also clear that the class $\mathcal{F}_1$ is a Γ_1-invariant σ-algebra of subsets of the space E_1. The functional ψ defined by

$$(\forall X)(X \in \mathcal{F}_1 \Rightarrow \psi(X) = \mu(X \times E_2))$$

is a Γ_1-invariant probability measure such that $Y_1 \in dom(\psi)$. This contradicts the Γ_1-absolute nonmeasurability of Y_1, and Theorem 5.16 is proved. $\square$

Finally, we focus on one result obtained in [85].

Theorem 5.17 *In the Euclidean space* $\mathbf{R}^n$ *there exists an* $\mathbf{R}^n$*-absolutely nonmeasurable subset.*

Using results of theorems 5.16 and 5.17, we can obtain the validity of the following proposition.

Theorem 5.18 *In the infinite-dimensional vector space* $\mathbf{R}^N$ *there exists an* $\mathbf{R}^{(N)}$*-absolutely nonmeasurable subset.*

Chapter 6

On Quasiinvariant Radon Measures in $\mathbf{R}^I$

Let us consider some definitions from measure theory. Assume that (X,τ) is a topological space, $\mathcal{B}(X)$ is the σ-algebra of all Borel subsets of the space X, generated by the topology τ. Let μ be an arbitrary measure defined on the σ-algebra $\mathcal{B}(X)$.

Definition 6.1 A Borel σ-finite measure μ is called regular if

$$(\forall B)(B \in \mathcal{B}(X) \ \& \ 0 \leq \mu(B) < +\infty \rightarrow \mu(B) = \sup_{\substack{K \subseteq X \\ \text{K is closed in X}}} \mu(K)).$$

Definition 6.2 A Borel σ-finite measure μ is called Radon if

$$(\forall B)(B \in \mathcal{B}(X) \ \& \ 0 \leq \mu(B) < +\infty \rightarrow \mu(B) = \sup_{\substack{K \subseteq X \\ \text{K is compact in X}}} \mu(K)).$$

Example 6.1 Let I be an arbitrary nonempty set of parameters. Let λ be the canonical Gaussian Borel measure on $\mathbf{R}^I$. On the one hand, we have that λ is a regular Borel probability measure quasiinvariant with respect to the everywhere dense vector subspace $\mathbf{R}^{(I)}$ of the space $\mathbf{R}^I$ (cf.Corollary 4.1). On the other hand, we have that the measure λ is not Radon when $Card(I) > \aleph_0$.

Let us consider a separable Banach space ℓ_1 of all absolutely summable real-valued sequences, defined on N. Let $\mathcal{B}(\ell_1)$ be a Borel σ-algebra of subsets of ℓ_1. By using Lemma 5.1 and Theorem 5.1 one can prove the validity of the following assertion

Lemma 6.1 *There exists a nontrivial σ-finite Borel measure μ_0 on ℓ_1 such that:*
1) $\mu_0(\prod_{i \in N}[0; \frac{1}{2^{i+1}}[) = 1$;
2) the vector space G_0 of all admissible translations (in the sense of invariance) of the measure μ_0 has the following form

$$G_0 = \{(a_k)_{k \in N} \ : \ (a_k)_{k \in N} \in \ell_1 \ \& \ \textit{the series} \sum_{k \in N} |a_k| \cdot 2^{k+1} \ \textit{is convergent}\}.$$

The following proposition is valid

Lemma 6.2 *Let us define an operator*

$$\varphi : \ell_1 \to C[0;1]$$

by the formula

$$(\forall (a_k)_{k\in N})((a_k)_{k\in N} \in \ell_1 \to \varphi((a_k)_{k\in N}) = \sum_{k\in N} a_k \cdot x^k).$$

Then an operator φ *is one-to-one linear continuous operator.*

The proof of this result is not difficult, and we leave it to the reader.

The above results immediately imply the following statement.

Theorem 6.1 *Let us denote by* μ_1 *a functional defined by the following formula*

$$(\forall B)(B \in \mathcal{B}(C[0;1]) \to \mu_1(B) = \mu_0(\varphi^{-1}(B))),$$

where $\mathcal{B}(C[0;1])$ *denotes an usual Borel* σ*-algebra of subsets of the separable Banach space* $C[0;1]$. *Then functional* μ_1 *is a such nontrivial* σ*-finite Borel measure that a vector space* G_1 *of all admissible translations (in the sense of invariance) has the following form*

$$G_1 = \{ f \mid f \in C[0;1] \;\&\; (\exists a_f = (a_k)_{k\in N} \in G_0) \;\&$$
$$(\forall x)(x \in [0;1] \to f(x) = \sum_{k\in N} a_k \cdot x^k)\}.$$

Remark 6.1 *Since* μ_0 *is* $R^{(N)}$*-invariant, we easily deduce that the vector space of all real-valued polynomials defined on* $[0;1]$ *is a vector subspace of* G_1.

By using Theorem 6.1 and a natural embedding of $C[0,1]$ in $R^{[0,1]}$, one can easily obtain the validity of the following assertion.

Theorem 6.2 *Let* $\mathbf{R}^{[0;1]}$ *be a vector space of all real-valued functions on* $[0;1]$, *equipped with Tykhonoff topology. Then there exists a such nontrivial* σ*-finite Radon measure* μ *on* $\mathbf{R}^{[0;1]}$, *which is invariant with respect to everywhere dense vector subspace of all polynomials on* $[0;1]$.

Remark 6.2 This fact that a vector subspace of all admissible translations of the measure μ is everywhere dense in $\mathbf{R}^{[0;1]}$ is a simple consequence of the classical Weierstrass theorem on approximation.

Example 6.2 Let λ_1 be an arbitrary probability Borel measure defined on the measurable space $(\mathbf{R}^{[0;1]}, \mathcal{B}(\mathbf{R}^{[0;1]}))$ and equivalent to the measure μ. It is clear that this measure is quasiinvariant under the group of all polynomials on $[0;1]$. It can be considered as a continuous image of some Radon measure defined on some Hausdorff topological space.

The following assertion is valid.

Theorem 6.3 *Assume that the set* I *satisfies the condition* $card(I) > \aleph_0$. *Then in the measurable space* $(\mathbf{R}^I, \mathcal{B}(\mathbf{R}^I))$ *there does not exist a Radon probability measure which would be quasiinvariant with respect to the vector space* $\mathbf{R}^{(I)}$.

The proof of this theorem can be obtained by assuming the contrary. Then we obtain a contradiction to the property of σ-finiteness of the measure.

Let us recall the following definition.

Definition 6.3 Assume that Γ is a family of real functions defined on the set X. A family Γ is called separating the points of the set X if

$$(\forall x_1)(\forall x_2)(x_1 \in X \ \& \ x_2 \in X \ \& \ x_1 \neq x_2 \to (\exists f)(f \in \Gamma \to f(x_1) \neq f(x_2))).$$

Lemma 6.3 *If $card(X) > c$, then an arbitrary countable family Γ of real functions defined on the set X does not separate the points of the set X.*

Proof. Indeed, to an arbitrary point $x \in X$ let us put into the correspondence a point $y(x) \in R^N$ such that

$$y(x) = (f_1(x), f_2(x), \cdots),$$

where $\Gamma = \{f_k\}_{k \in N}$. Since card$(\mathbf{R}^N) = c$, there exists a subset $Y \subseteq X$ such that

$$(a) \ \mathrm{card}(Y) = \mathrm{card}(X),$$

$$(b) \ (\forall z)(\forall x)(z \in Y \ \& \ x \in Y \to y(x) = y(z)).$$

If $x_1 \in Y$, $x_2 \in Y$ and $x_1 \neq x_2$, then there is no function $f \in \Gamma$ such that

$$f(x_1) \neq f(x_2).$$

Lemma 6.3 is proved. $\square$

Now we can formulate and prove the main result of the present chapter.

Theorem 6.4 *If I is a set of indices with $card(I) > c$ (the cardinality of the continuum is denoted by c), then in the measurable space $(\mathbf{R}^I, \mathcal{B}(R^I))$ there exists no Radon probability measure which would be quasiinvariant with respect to some everywhere dense vector subspace of $\mathbf{R}^I$.*

Proof. Assume the contrary and let μ be a Radon probability measure on the measurable space $(\mathbf{R}^I, \mathcal{B}(\mathbf{R}^I))$ which is quasiinvariant with respect to some everywhere dense in $\mathbf{R}^I$ vector subspace $G \subseteq \mathbf{R}^I$. This means that, for some compact subset K of $\mathbf{R}^I$, we have

$$0 < \mu(K) < +\infty.$$

Then there exists a countable family $(g_n)_{n \in N}$ of elements of the group G such that $\bigcup_{n \in N}(K + g_n)$ is a μ-almost G-invariant subset of the space $\mathbf{R}^I$, i.e.,

$$(\forall h)(h \in G \to \mu((\bigcup_{n \in N}(K + g_n)) \triangle ((\bigcup_{n \in N}(K + g_n)) + h) = 0).$$

Using Lemma 6.3, we obtain that the family $(g_n)_{n \in N}$ does not separate the set I. Therefore there exist two different points $i_1 \in I$ and $i_2 \in I$ such that

$$(\forall n)(n \in N \to g_n(i_1) = g_n(i_2)).$$

Let us put
$$M=\operatorname{diam}(P_{r_{i_1}}(K)\cup\{0\}).$$
So G is everywhere dense, for some $h^*\in G$ we have
$$h^*(i_1)\in]g_1(i_1)+2M;g_1(i_1)+3M[.$$
Clearly,
$$\mu((\bigcup_{n\in N}(K+g_n)+h^*)\setminus\bigcup_{n\in N}(K+g_n))>0,$$
which contradicts the condition
$$(\forall h)(h\in G\rightarrow\mu(((\bigcup_{n\in N}(K+g_n))+h)\triangle\bigcup_{n\in N}(K+g_n))=0).$$
Thus, Theorem 6.4 is proved. □

Chapter 7

On Partial Analogs of the Lebesgue Measure

The problem of the existence of an partial analog of the Lebesgue measure on infinite-dimensional topological vector space itself is interesting and it is so important that has been studied for more than a half century ago by many people using various approaches (cf.[50],[170] and others). Among them the result of I.V.Girsanov and B.S. Mityagin [50] should be mentioned especially. Their result asserts that an arbitrary σ-finite quasi-invariant Borel measure defined on infinite-dimensional locally convex topological vector space is identically zero. This result asserts that the properties of the σ-finiteness and the translation-invariance are not consistent for non-zero Borel measures in infinite-dimensional topological vector spaces. A.B. Kharazishvili [87] ignoring the property of translation-invariance, constructed an example of a such non-zero σ-finite Borel measure in the Hilbert space ℓ_2 which is invariant with respect to everywhere dense (in ℓ_2) linear manifold(cf. Lemma 5.1 and Remark 5.3). In the present chapter we ignore the property of the σ-finiteness and give a construction of non-zero non-atomic translation-invariant Borel measures on infinite-dimensional separable Banach spaces with a basis in the well-known Solovay model (cf.[167]) which is the following system of axioms:

$$(ZF)\&(DC)\&(\text{every subset of } \mathbf{R} \text{ is measurable in the Lebesgue sense}),$$

where (ZF) denotes the Zermelo-Fraenkel set theory and (DC) denotes the axiom of Dependent Choices (cf. Chapter 1).

We focus on the classes of null sets generated by partial analogs of Lebesgue measure and study their relations with respect to various σ-ideals of null sets introduced in the papers [29],[30],[70],[146].

Now, let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be two measure spaces. The measures μ_1 and μ_2 are called isomorphic if there exists a measurable isomorphism from E_1 onto E_2 such that

$$(\forall X)(X \in S_1 \rightarrow \mu_1(X) = \mu_2(f(X))).$$

In the sequel we need some auxiliary lemmas.

Lemma 7.1 *Let E_1 and E_2 be any two Polish topological spaces. Let μ_1 be a probability diffused Borel measure on E_1 and let μ_2 be a probability diffused Borel measure on E_2. Then*

there exists a Borel isomorphism $\varphi : (E_1, B(E_1)) \to (E_2, B(E_2))$ *such that*

$$\mu_1(X) = \mu_2(\varphi(X))$$

for every $X \in B(E_1)$.

The proof of Lemma 7.1 can be found in [25].

Lemma 7.2 *Let* E *be a Polish space and let* μ *be a probability diffused Borel measure on* E. *Then, in Solovay's model the completion* $\overline{\mu}$ *of* μ *is defined on the power set of* E.

Proof. Let b_1 be the standard Lebesgue measure on $[0,1]$. According to Lemma 7.1, the measure b_1 is Borel isomorphic to the probability measure μ. Denote this isomorphism by φ and consider an arbitrary set $W \subset E$. It is clear that $\varphi^{-1}(W) = X \cup Y$, where $X \in B([0,1])$ and $(\exists Z)(Y \subset Z \in B([0,1])\ \&\ b_1(Z) = 0)$. Let us take the set $X \cup Z$. The isomorphism between the measures b_1 and μ implies $\mu(\varphi(Z \setminus X)) = 0$. On the one hand, we can write $W = \varphi(X) \cup \varphi(Y)$. On the other hand, we have $\varphi(Y) \subset \varphi(Z)$. Clearly, $\overline{\mu}(\varphi(Y)) = 0$ since $\mu(\varphi(Z)) = 0$. $\square$

Corollary 7.1 *Let* $\mathbb{J}$ *be any non-empty subset of the set all natural numbers* $\mathbb{N}$. *Let, for* $k \in \mathbb{J}$, S_k *be the unit circle in the Euclidean plane* $\mathbf{R}^2$. *We may identify the circle* S_k *with a compact group of all rotations of* $\mathbf{R}^2$ *about its origin. Let* $\lambda_{\mathbb{J}}$ *be the probability Haar measure defined on the compact group* $\prod_{k\in\mathbb{J}} S_k$. *Then in Solovay's model the completion* $\overline{\lambda_{\mathbb{J}}}$ *of* $\lambda_{\mathbb{J}}$ *is defined on the power set of* $\prod_{k\in\mathbb{J}} S_k$.

Remark 7.1 For $k \in \mathbb{J}$, define the function f_k by $f_k(x) = \exp\{2\pi x i\}$ for every $x \in \mathbf{R}$. Then the equality

$$(\prod_{k\in\mathbb{J}} f_k)(z+w) = (\prod_{k\in\mathbb{J}} f_k)(z) \circ (\prod_{k\in\mathbb{J}} f_k)(w)$$

holds for every $z, w \in \mathbf{R}^{\mathbb{J}}$, where $\mathbf{R}^{\mathbb{J}}$ denotes the vector space of all real-valued sequences defined on $\mathbb{J}$, $\prod_{k\in\mathbb{J}} f_k$ denotes the direct product of the family of functions $(f_k)_{k\in\mathbb{J}}$, "$\circ$" denotes the group operation in $\prod_{k\in\mathbb{J}} S_k$.

Remark 7.2 For $E \subset \mathbf{R}^{\mathbb{J}}$ and $g \in \prod_{k\in\mathbb{J}} S_k$, put

$$f_E(g) = \begin{cases} \mathrm{card}((\prod_{k\in\mathbb{J}} f_k)^{-1}(g) \cap E), & \text{if this is finite;} \\ +\infty, & \text{in all other cases.} \end{cases}$$

Then

$$(\forall h)(h = (h_k)_{k\in\mathbb{J}} \in \mathbf{R}^{\mathbb{J}} \Rightarrow f_{E+h}(g) = f_E(g \circ g_h)),$$

where

$$g_h = (\prod_{k\in\mathbb{J}} f_k)(-h) = (\exp\{-2\pi h_k i\})_{k\in\mathbb{J}}.$$

Theorem 7.1 *Let* $\mathbb{J}$ *be any non-empty subset of the set all natural numbers* $\mathbb{N}$. *Then, in Solovay's model there exists a translation-invariant measure* $\mu_{\mathbb{J}}$ *on the powerset* $\mathbf{R}^{\mathbb{J}}$ *such that* $\mu_{\mathbb{J}}([0;1]^{\mathbb{J}}) = 1$.

Proof. Define the functional μ by

$$(\forall E)(E \subset \mathbf{R}^{\mathbb{J}} \rightarrow \mu_{\mathbb{J}}(E) = \int\limits_{\prod_{k\in\mathbb{J}} S_k} f_E(g) d\overline{\lambda_{\mathbb{J}}}(g)).$$

The functional $\mu_{\mathbb{J}}$ is a measure since $f_{\emptyset}(g) = 0$ and

$$f_{\cup_{k\in\mathbb{N}} E_k}(g) = \sum_{k\in\mathbb{N}} f_{E_k}(g),$$

where $(E_k)_{k\in\mathbb{N}}$ is an arbitrary family of disjoint subsets of $\mathbf{R}^{\mathbb{N}}$.

The same functional μ is a translation-invariant measure. Indeed, for an arbitrary $h \in \mathbf{R}^{\mathbb{J}}$, by the invariance of the measure $\overline{\lambda_{\mathbb{J}}}$ and by Remark 7.2, we have

$$\mu_{\mathbb{J}}(E+h) = \int\limits_{\prod_{k\in\mathbb{J}} S_k} f_{E+h}(g) d\overline{\lambda_{\mathbb{J}}}(g) =$$

$$= \int\limits_{\prod_{k\in\mathbb{J}} S_k} f_E(\Phi(g)) d\overline{\lambda_{\mathbb{J}}}(g) = \int\limits_{\Phi^{-1}(\prod_{k\in\mathbb{J}} S_k)} f_E(\Phi(g)) d\overline{\lambda_{\mathbb{J}}}(g) =$$

$$= \int\limits_{\prod_{k\in\mathbb{J}} S_k} f_E(g) d\overline{\lambda_{\mathbb{J}}}_{\Phi}(g) = \int\limits_{\prod_{k\in\mathbb{J}} S_k} f_E(g) d\overline{\lambda}(g) = \mu_{\mathbb{J}}(E),$$

where we use the transformation rule for the image $\overline{\lambda_{\mathbb{J}}}_{\Phi}$ of $\overline{\lambda_{\mathbb{J}}}$ with respect to the transformation $\Phi : x \rightarrow x \circ \overline{h}$.

Note that

$$(\forall g)(g \in \prod_{k\in\mathbb{J}} S_k \rightarrow f_{[0;1[^{\mathbb{J}}}(g) = 1).$$

This implies that $\mu_{\mathbb{J}}([0;1[^{\mathbb{J}}) = 1$.

It is clear that

$$[0;1]^{\mathbb{J}} \setminus [0;1[^{\mathbb{J}} = \cup_{k\in\mathbb{J}} X_k,$$

where $X_k = \{1\}_k \times [0;1]^{\mathbb{J}\setminus\{k\}}$ and $\mu_{\mathbb{J}}(X_k) = 0$. $\square$

Remark 7.3 Note that the measure $\mu_{\mathbb{J}}$ coincides with the standard Lebesgue measure on $\mathbf{R}^{\mathbb{J}}$ for every finite $\mathbb{J}$. By this reason, the measure $\mu_{\mathbb{N}}$ can be regarded as a partial analog of Lebesgue measure on $\mathbf{R}^{\mathbb{N}}$.

Let us consider the family of conditional measures

$$(\mu_{\mathbb{N}}(\cdot|[0,1]^{\mathbb{N}} + a))_{a\in\mathbf{R}^{\mathbb{N}}}.$$

We say that $X \subset \mathbf{R}^{\mathbb{N}}$ is a standard cub null set in $\mathbf{R}^{\mathbb{N}}$, if

$$\mu_{\mathbb{N}}(X|[0,1]^{\mathbb{N}} + a) = 0$$

for every $a \in \mathbf{R}^{\mathbb{N}}$. The class of all standard cub null sets in $\mathbf{R}^{\mathbb{N}}$ we denote by $\mathbb{S}.\mathbb{C}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}})$.

We say that $X \subset \mathbf{R}^{\mathbb{N}}$ is a standard Lebesgue null set in $\mathbf{R}^{\mathbb{N}}$, if $\mu_{\mathbb{N}}(X) = 0$. We denote the class of all subsets (in $\mathbf{R}^{\mathbb{N}}$) of $\mu_{\mathbb{N}}$-measure zero by $\mathbb{S}.\mathbb{L}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}})$.

Theorem 7.2 $\mathbb{S}.\mathbb{L}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}}) \subset \mathbb{S}.\mathbb{C}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}})$.

Proof. One can easily demonstrate that $\mathbb{S}.\mathbb{L}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}}) \subseteq \mathbb{C}.\mathbb{N}.\mathbb{S}.(\mathbf{R}^{\mathbb{N}})$.
Now, let $\underline{z} = (z_1, z_2, \cdots) \in [0,1[^{\mathbb{N}}$.
We have
$z_1 = 0, a_1^{(1)} \, a_2^{(1)} \, \cdots$
$z_2 = 0, a_1^{(2)} \, a_2^{(2)} \, \cdots$
$z_3 = 0, a_1^{(3)} \, a_2^{(3)} \, \cdots$
$\vdots$,
where $0 \leq a_n^{(m)} \leq 9$ for every $n, m \in \mathbb{N}$.
Put
$f^{(1)}(\underline{z}) = z_1 + 2a_1^{(1)}$,
$f^{(2)}(\underline{z}) = z_2 + 2a_2^{(1)}$,
$f^{(3)}(\underline{z}) = z_3 + 2a_1^{(2)}$,
$f^{(4)}(\underline{z}) = z_4 + 2a_1^{(3)}$,
$f^{(5)}(\underline{z}) = z_5 + 2a_2^{(2)}, \cdots$.
We set

$$F = \{(f^{(k)}(\underline{z}))_{k \in \mathbb{N}} : \underline{z} \in [0,1[^{\mathbb{N}}\}.$$

Let us show that F is a cub null set in $\mathbf{R}^{\mathbb{N}}$. In this direction it suffices to show that

$$(\forall a)(a \in \mathbf{R}^{\mathbb{N}} \rightarrow \mathrm{card}(F \cap ([0,1]^{\mathbb{N}} + a)) \leq 1).$$

Assume the contrary and let for different elements $\underline{z_1}, \underline{z_2} \in [0,1[^{\mathbb{N}}$ we have $f(\underline{z_i}) \in [0,1]^{\mathbb{N}} + a$ for $i = 1,2$. It means that

$$(\forall k)(k \in \mathbb{N} \rightarrow |f^{(k)}(\underline{z_1}) - f^{(k)}(\underline{z_2})| \leq 1).$$

As $\underline{z_1} \neq \underline{z_2}$, there exists $k_0 \in \mathbb{N}$ that $z_{k_0}^{(1)} \neq z_{k_0}^{(2)}$. Let $z_{k_0}^{(1)} = 0, a_1^{(k_0)} \, \cdots$ and $z_{k_0}^{(2)} = 0, b_1^{(k_0)} \, \cdots$. Using the same argument there exists $m_0 \in \mathbb{N}$ such that

$$|a_{m_0}^{(k_0)} - b_{m_0}^{(k_0)}| \geq 1.$$

For some $n_0 \in \mathbb{N}$ we have
$f^{(n_0)}(\underline{z_1}) = x_1 + 2a_{m_0}^{(k_0)}$,
$f^{(n_0)}(\underline{z_2}) = x_2 + 2b_{m_0}^{(k_0)}$,
where $0 \leq x_1 < 1, \ 0 \leq x_2 < 1$.
We have

$$|f^{(n_0)}(\underline{z_1}) - f^{(n_0)}(\underline{z_2})| = |x_1 + 2a_{m_0}^{(k_0)} - x_2 - 2b_{m_0}^{(k_0)}| =$$

$$|2(a_{m_0}^{(k_0)} - b_{m_0}^{(k_0)}) - (x_2 - x_1)| \geq |2(a_{m_0}^{(k_0)} - b_{m_0}^{(k_0)})| - |x_2 - x_1| > 1.$$

The last relation means that F is a cub null set in $\mathbf{R}^{\mathbb{N}}$. On the other hand, we have

$$\mu_{\mathbb{N}}(F) = \int_{\prod_{k\in\mathbb{N}} S_k} f_F(g) d\overline{\lambda_{\mathbb{N}}}(g).$$

So $f_F(g) = 1$ for every $g \in \prod_{k\in\mathbb{N}} S_k$, we conclude that $\mu_{\mathbb{N}}(F) = 1$. This ends the proof of Theorem 7.2. □

Here we are going to give a construction of partial analogs of the Lebesgue measure in infinite-dimensional separable Banach spaces with basis.

Let B be an infinite-dimensional separable Banach space with basis.

We say that a transformation $A : B \to R^N$ belongs to a class $\mathcal{A}$, if there exists a basis $e_1, e_2, \cdots$ in B such that $\sum_{i\in\mathbb{N}} ||e_i|| < \infty$ and $A(z) = (x_k)_{k\in\mathbb{N}}$, where $z = \sum_{k\in N} x_k e_k$.
For $A \in \mathcal{A}$, we set $\mu_A(X) = \mu_{\mathbb{N}}(A(X))$ for $X \subset B$.

One can easily establish the validity of the following assertion.

Theorem 7.3 *For every $A \in \mathcal{A}$, the measure μ_A is a translation-invariant Borel measure defined on the powerset $\mathcal{P}(B)$ of B which gets a numerical value one on the compact set Δ, where $\Delta = A^{-1}([0,1]^N)$*

Proof. Indeed, for $h \in B$ and $X \subset B$, we have

$$\mu_A(X+h) = \mu_N(A(X+h)) = \mu_N(A(X)+A(h)) = \mu_N(A(X)) = \mu_A(X).$$

Obviously,

$$\mu_A(\Delta) = \mu_N(A(A^{-1}([0,1]^N))) = \mu_N([0,1]^N) = 1.$$

This ends the proof of Theorem 7.3.

Now we discuss the problem which stimulates extending the measure theoretic terms "measure zero" and "almost every" in infinite-dimensional Banach spaces. Let B be any infinite-dimensional Banach space, let P be any sentence formulated for elements in B and let μ be any probability Borel measure on B. One can easily discover that an information contained in the following sentence:

"μ-almost every element of B satisfies the property P",

in general, is very poor, because the measure μ is concentrated on the union of countable compact subsets $(F_k)_{k\in N}$ in B (cf.[86]) and for arbitrary $k \in N$ there exists a vector $v_k \in B$ which spans a line L_k such that every translation of L_k meets F_k in at most one point(cf.[70]),p.225,Fact 8). In other words, the support of μ may be regarded as an union of countable family of "surfaces".

We remind the reader of some notions of classes of null sets which will be under consideration. The reader can also see many interesting applications of introduced here notions of null sets in various articles (see,e.g., [29],[30],[70],[98],[100],[113],[115],[146],[148],[168] and so on).

Let B be again an infinite-dimensional separable Banach space and $e_1, e_2, \cdots \in B$ be an arbitrary element of B that $\sum_{i \in \mathbb{N}} ||e_i|| < \infty$ and the span of $e_1, e_2, \cdots$ is dense in B.

Definition 7.1 A set $X \subset B$ is called an Aronszajn null set in B(cf.[30]), if it can be written as a union of Borel sets E_n such that each E_n is null on every line in the direction e_n(i.e., for every $a \in B,\ \mu_1(\{t \in \mathbf{R} : a + te_n \in E_n\}) = 0$, where μ_1 is the one-dimensional standard Lebesgue measure).

Definition 7.2 Following Mankiewicz (cf.[30]), a Borel set $X \subset B$ is called a cube null set, if it is null for every non-degenerate cube measure (Non-degenerate cube measures in B may be defined as distributions of the random variable of the form $a + \sum_{k \in \mathbb{N}} X_k e_k$, where $a \in B$,$(X_k)_{k \in N}$ are uniformly distributed mutually independent random variables with values in $[0,1]$ and $e_1, e_2, \cdots \in B$ is an arbitrary element of B that $\sum_{i \in \mathbb{N}} ||e_i|| < \infty$ and the span of $e_1, e_2, \cdots$ is dense in B.

Definition 7.3 A Borel set $X \subset B$ is called a cube null set in basis $\Gamma = (e_k)_{k \in N}$, if it is null for every non-degenerate cube measure defined by the basis Γ.

Definition 7.4 Let K be the class of all non-zero finite measures defined on the Borel σ-field $\mathcal{B}(B)$. We denote by $\mathcal{B}(B)^{\mu}$ the completion of $\mathcal{B}(B)$ with respect to the measure μ for $\mu \in K$. A set $E \subset B$ called universally measurable if $E \in \cap_{\mu \in K} \mathcal{B}(B)^{\mu}$.

Remark 7.4 Using Lemma 7.2, one can easily demonstrate that in Solovay's model every subset in an arbitrary Polish topological vector space is universally measurable.

Definition 7.5 Following Christensen [29], a universally measurable set E is Haar null if there is a Borel probability measure μ on B that every translation of E has $\overline{\mu}$-measure zero, where $\overline{\mu}$ denotes an usual completion of the measure μ.

Definition 7.6 Following Brian R. Hurt, Tim Sauer and James A. Yorke (cf. [70]), a set X is called shy if it is a subset of a Borel set X' for which $\mu(X' + v) = 0$ for every $v \in B$ and some Borel probability measure μ such that $\mu(K) = \mu(B)$ for some compact K.

Remark 7.5 Using Lemma 7.2, one can easily demonstrate that in Solovay's model every subset in a separable Banach space B is universally measurable. Hence, the notions of expanding Haar null sets and Haar null sets (as well the notion of expanding shy sets) coincide in the Solovay model for infinite-dimensional separable Banach spaces.

Definition 7.7 Following Phelps (cf.[148]), a Borel set is called Gaussian null set if it is null for every Gaussian measure on B(A Gaussian measure in B may be defined as a distribution of a.s. convergent sums $a + \sum_{k \in N} X_k e_k$, where $a \in B$ and $(X_k)_{k \in N}$ are mutually independent standard Gaussian variables).

Definition 7.8 We say that $X \subset B$ is a quasi-Lebesgue null set in B if $\overline{\mu}_A(X) = 0$ for some $A \in \mathcal{A}$, where $\overline{\mu}_A$ denotes an usual completion of the measure μ_A.

Definition 7.9 We say that $X \subset B$ is a Lebesgue null set in B if $\overline{\mu}_A(X) = 0$ for arbitrary $A \in \mathcal{A}$.

Definition 7.10 A measure μ on E is called quasi-finite if there exists $Y_0 \subseteq E$ that $0 < \mu(Y_0) < \infty$.

Definition 7.11 Let $\Gamma = (e_i)_{i \in N}$ be a basis in B that $\sum_{i \in N} ||e_i|| < \infty$. We say that $X \subset B$ is cube null set in basis Γ, if $\mu_A(X) = 0$, where $A(z) = (x_k)_{k \in N}$ for $z = \sum_{k \in N} x_k e_k$.

Definition 7.12 We say that a set $X \subset B$ is a quasi-finite translation null set if there exists a quasi-finite translation-invariant Borel measure ν such that $\overline{\nu}(X) = 0$, where $\overline{\nu}$ denotes the usual completion of the measure ν.

Definition 7.13 We say that $X \subset B$ is a translation null set in B if there exists a non-zero translation-invariant Borel measure μ and a Borel set S' such that $S \subseteq S'$ and $\mu(S') = 0$.

We denote by

$\mathbb{L}.\mathbb{N}.\mathbb{S}(B), \mathbb{L}.\mathbb{N}.\mathbb{S}(B,\Gamma), \mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B), \mathbb{G}.\mathbb{N}.\mathbb{S}.(B),$
$\mathbb{A}.\mathbb{N}.\mathbb{S}(B), \mathbb{C}.\mathbb{N}.\mathbb{S}.(B), \mathbb{C}.\mathbb{N}.\mathbb{S}.(B,\Gamma), \mathbb{H}.\mathbb{N}.\mathbb{S}.(B), \mathbb{S}.\mathbb{S}.(B), \mathbb{T}.\mathbb{N}.\mathbb{S}.(B)$

classes of Lebesgue null sets, Lebesgue null sets in the basis Γ, quasi-finite translation null sets, Gaussian null sets, Aronszajn null sets, cube null sets, cube null sets in the basis Γ, Haar null sets, shy sets and translation null sets in B, respectively.

For every infinite-dimensional separable Banach space B the following relations

$$\mathbb{G}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{A}.\mathbb{N}.\mathbb{S}(B) = \mathbb{C}.\mathbb{N}.\mathbb{S}.(B) \subset \mathbb{H}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{S}.\mathbb{S}.(B)$$

are valid.

The result

$$\mathbb{H}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{S}.\mathbb{S}.(B)$$

is due to Maxwell B. Stinchombe(cf.[168]).

The assertion

$$\mathbb{C}.\mathbb{N}.\mathbb{S}.(B) \subset \mathbb{H}.\mathbb{N}.\mathbb{S}.(B)$$

was established by Y.Benyamini and J. Linderstrauss (cf.[30]).

The validity of the following equalities

$$\mathbb{G}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{A}.\mathbb{N}.\mathbb{S}(B) = \mathbb{C}.\mathbb{N}.\mathbb{S}.(B)$$

was proved by Marianna Csörnyei (cf.[30]).

We have the following propositions in Solovay's model.

Theorem 7.4. *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma = (e_k)_{k \in N}$ such that $\sum_{k \in N} ||e_k|| < \infty$. Then in Solovay's model we have*

$$\mathbb{L}.\mathbb{N}.\mathbb{S}.(B) \subset \mathbb{L}.\mathbb{N}.\mathbb{S}.(B,\Gamma) \subset \mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B).$$

Theorem 7.5. *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma = (e_k)_{k \in N}$ such that $\sum_{k \in N} ||e_k|| < \infty$. Then in Solovay's model we have*

$$\mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B) \subseteq \mathbb{S}.\mathbb{S}(B).$$

Proof. Let $X \in \mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B)$. It means that there exists a quasifinite translation-invariant measure μ defined on the powerset of B such that $\mu(X)=0$. Since μ is a quasifinite there exists a compact set K in B such that $0<\mu(K)<\infty$. We have

$$(\forall h)(\forall Y)(h \in B \,\&\, Y \subseteq B \rightarrow \mu(Y+h|K) \leq \mu(Y+h)=0),$$

where $\mu(\cdot|K)$ denotes a conditional probability measure, defined by

$$(\forall Y)(\, Y \subseteq B \rightarrow \mu(Y|K)=\frac{\mu(Y \cap K)}{\mu(K)}).$$

Note that the measure $\mu(\cdot|K)$ is a completion of its restriction to the Borel class of B.
Hence, the validity of the relation

$$(\forall h)(h \in B \ \rightarrow \mu(X+h|K) \leq \mu(X+h)=0)$$

implies that $X \in \mathbb{S}.\mathbb{S}(B)$. $\square$

Theorem 7.6. *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma=(e_k)_{k \in N}$ such that $\sum_{k \in N}||e_k||<\infty$. Then in Solovay's model we have*

$$\mathbb{L}.\mathbb{N}.\mathbb{S}(B,\Gamma) \subseteq \mathbb{C}.\mathbb{N}.\mathbb{S}(B,\Gamma).$$

In this context note that there does not exist a probability Borel measure p on $\mathbb{B}$ (cf. Theorem 10.1) such that

$$(\forall X)(\overline{p}(X)=0 \rightarrow X \in \mathbb{S}.\mathbb{S}(\mathbb{B})).$$

Let μ be a quasifinite measure defined on E and let $f: E \rightarrow \overline{R^+}$ be a non-negative μ-measurable function. A number $a \in \overline{R^+}$ defined by

$$a=\sup\{\int_{B_1} f(x)d(\mu): B_1 \subset B,\ 0<\mu(B_1)<\infty\}$$

is called μ-integral of f and is denoted by $\int_B f(x)d(\mu)(x)$. If a is finite then f is called μ-integrable.

Analogously, let μ and ν be quasifinite measures defined on E and let $g: E \times E \rightarrow \overline{R^+}$ be a nonnegative $\mu \times \nu$-measurable function. Then a number $b \in \overline{R^+}$ defined by

$$b=\sup\{\int_{B_1 \times B_2} g(x,y)d(\mu \times \nu)(x,y):\ 0<\mu(B_1)\cdot\mu(B_2)<\infty\}$$

is called $\mu \times \nu$-integral of g and is denoted by $\int\int_{E \times E} g(x,y)d(\mu \times \nu)(x,y)$. If b is finite then g is called $\mu \times \nu$-integrable.

The following result is valid.

Lemma 7.3 *Let μ and ν be quasifinite Borel measures defined on B. Let $S \subset B$ be a Borel subset such that a characteristic function Ind_{S^Σ} of the set $S^\Sigma=\{(x,y) \in B \times B: x+y \in S\}$ is $\mu \times \nu$-integrable, then*

$$\int_{B \times B} Ind_{S^\Sigma}(x,y)d(\mu \times \nu)(x,y)=\int_B \mu(S-y)d(\nu)(y)=\int_B \nu(S-x)d(\mu)(x).$$

As a corollary of Lemma 7.3, we obtain the following:

Theorem 7.7 *Let S be a Borel set in B. Let there exist a probability measure P with compact support such that every translation of S has p-measure zero(i.e., $S \in \mathbb{S}.\mathbb{S}.(B)$). If μ is a quasifinite translation-invariant Borel measure in B such that $Ind_{S\Sigma}$ is $p \times \mu$-integrable then $\mu(S) = 0$.*

Proof. Applying the result of Lemma 7.3, the property of shyness of S and the property of translation-invariance of μ, we get

$$0 = \int_{B\times B} \mathrm{Ind}_{S\Sigma}(x,y)d(p\times\mu)(x,y)$$

$$= \int_B p(S-y)d(\mu)(y) = \int_B \mu(S-x)d(p)(x) = \mu(S)\times p(B) = \mu(S). \quad \square$$

The following proposition is valid.

Theorem 7.8 $\mathbb{S}.\mathbb{S}.(B) \subseteq T.N.S.(B)$.

Remark 7.6 The simple proof of Theorem 7.8 can be obtained by the scheme presented in [70] (cf. p.219): let us define a measure μ_p on Borel sets S by $\mu_p(S) = 0$ if every translation of S has p-measure zero and $\mu_p(S) = \infty$ otherwise.

Definition 7.14 We say that a finite-dimensional subspace $\Gamma \subset B$ is a probe for a set $T \subset B$ if

$$(\forall h)(h \in B \to \mu_\Gamma(S + h \cap \Gamma) = 0),$$

where μ_Γ denotes a Lebesgue measure supported on Γ.

The class of all subsets of B which are defined by an n-dimensional probe is defined by $n(B)$. It is clear that the class of all subsets of B which are defined by the finite-dimensional probes coincides with $\cup_{n\in N} n(B)$.

We have the following interesting result.

Theorem 7.9 *Let $S \in n(B)$. Then, in Solovay's model there exists a quasi-finite translation-invariant Borel measure μ on B such that $\mu(S) = 0$.*

Proof. Let Γ be n-dimensional vector subspace, μ_Γ be a Lebesgue measure supported on Γ and S' be a Borel subset of B containing S such that

$$(\forall h)(h \in B \to \mu_\Gamma(S' + h \cap \Gamma) = 0).$$

Let $(e_i)_{1\leq i\leq n}$ be a basis of Γ. Let $\Gamma^\perp$ be a cospace of Γ. Let $e_{n+1}, e_{n+2}, \cdots$ be such basis in $\Gamma^\perp$ that $\sum_{i>n} ||e_i|| < \infty$.

We set

$$A^\perp(z^\perp) = (x_k)_{k>n}$$

for $z^\perp = \sum_{k>n} x_k e_k \in \Gamma^\perp$.

Put

$$(\forall X)(X \subseteq \Gamma^\perp \to \mu_{\Gamma^\perp}(X) = \mu_{\mathbb{N}\setminus\{1,\cdots,n\}}(A^\perp(X)).$$

It is clear that $\mu_{\Gamma^{\perp}}$ is a translation-invariant measure defined on the power set of $\Gamma^{\perp}$ which gets a value one on the compact set

$$\Delta^{\perp} = \{\sum_{k>n} x_k e_k : 0 \leq x_k \leq 1 \ \& \ k \in \mathbb{N} \setminus \{1, \cdots, n\}\}.$$

Put

$$(\forall X)(X \subseteq B \rightarrow \mu(X) = \int_{\Gamma^{\perp}} \mu_{\Gamma}((X+h) \cap \Gamma) d\mu_{\Gamma^{\perp}}(h)).$$

It is not difficult to check that μ is a translation-invariant measure defined on the power set of B, which gets a value one on the compact set

$$\Delta = \{\sum_{k \geq 1} x_k e_k : 0 \leq x_k \leq 1 \ \& \ k \in \mathbb{N}\}.$$

Finally we have

$$\mu(S^{'}) = \int_{\Gamma^{\perp}} \mu_{\Gamma}(S^{'} + h \cap \Gamma) d\mu_{\Gamma^{\perp}}(h) = 0.$$

Since μ is the quasi-finite translation invariant measure defined on the powerset of B, using Theorem 7.5 we conclude that $n(B) \subseteq Q.T.N.S.(B)$. This ends the proof of Theorem 7.9 $\square$

As a consequence of theorems 7.5 and 7.9, we obtain the following:

Corollary 7.2 *Let X be an arbitrary Borel subset of the separable Banach space $(B, ||\cdot||)$ with basis, $y \in B$ and $||y|| = 1$. We say that x is a density point of X in direction y, if*

$$\lim_{h \rightarrow 0^{+}} \frac{\mu(X \cap [x - hy; x + yh])}{2 \cdot h} = 1,$$

where μ denotes one-dimensional Lebesgue measure concentrated on the line $L = \{x + ty : t \in R\}$ determined by the condition $\mu([x, x+y]) = 1$.

Let $(e_k)_{k \in N}$ be a basis in B such that $e_1 = y$, $\sum_{k \in \mathbb{N}} ||e_k|| < \infty$. Let Γ be a vector space generated by e_1. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem 7.9, we have

$$\mu(\{x \in X : x \textit{ is not a density point of } X \textit{ in direction } y\}) = 0.$$

Remark 7.7 It can be shown that the connection of Theorem 7.9 with the Solovay model is essential. In this direction we remind the reader of the notion of the uniform subset: a subset $X \subset B$ is called uniform if there exists a line L that every translate of L meets X in at most one point. Note that every compact in infinite-dimensional Polish topological vector space is uniform(cf.[70], p.225, Fact 8). The problem whether a uniform subset $X \subset R^n$ is shy is not solvable in ZF. Indeed, on the one hand, following theorems 7.5 and 7.9, in Solovay's model, for an arbitrary uniform set $X \subset R^n$ we have that

$$X \in 1(R^n) \subset Q.T.N.S.(R^n) \subset S.S.(R^n).$$

On the other hand, in the system of axioms

$$(ZF) \ \& \ (AC) \ \& \ (CH)$$

R^n can be represented as a union of countable family $(X_k)_{k\in N}$ of uniform subsets. Hence, there exists $k_0 \in N$ such that X_{k_0} is not measurable in the Lebesgue sense (note that every uniform subset $X \subset R^n$ measurable in the Lebesgue sense is of Lebesque measure zero) and consequently X_{k_0} is not shy.

Remark 7.8 Many properties in infinite-dimensional separable Banach spaces which hold "almost everywhere" in the sense of [70] can be proved by using finite-dimensional probes(see, e.g., [70],pp.226-227). Theorem 7.9 states that we can obtain the validity of all similar results in infinite-dimensional separable Banach spaces with basis by the technique of partial analogs of Lebesgue measure in Solovay's model. In particular, we have the following interesting interpretations of the below-mentioned well-known results.

Interpretation 7.1 ([70], Proposition 1) *Let $B = L^1[0,1]$. Let $(e_k)_{k\in\mathbb{N}}$ be a basis in $L^1[0,1]$ such that $e_1 = 1,\ \sum_{k\in\mathbb{N}} ||e_k||_{L^1[0,1]} < \infty$. Let Γ be a vector space generated by e_1. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem 7.9 we have that μ-almost every function f satisfies the condition $\int_0^1 f(x)dx \neq 0$.*

Interpretation 7.2 ([70], *Proposition* 2) *Let $B = l^p$ for $1 < p \leq \infty$. Let $(e_k)_{k\in\mathbb{N}}$ be a basis in l^p such that $e_1 = (\frac{1}{i})_{i\in\mathbb{N}}$ and $\sum_{k\in\mathbb{N}} ||e_k||_{l^p} < \infty$. Let Γ be a vector space generated by e_1. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem 7.9 we have that μ-almost every sequence $(a_k)_{k\in\mathbb{N}}$ deverges .*

Interpretation 7.3 ([70], *Proposition* 2) *Let $B = C[0,1]$ and let g and h be a pair of functions from $C[0,1]$ for which $\lambda g + \mu h$ is nowhere differentiable for every $(\lambda,\mu) \in \mathbf{R}^2$. Let $(e_k)_{k\in\mathbb{N}}$ be a basis in $C[0,1]$ such that $e_1 = g,\ e_2 = h$ and $\sum_{k\in\mathbb{N}} ||e_k||_{C[0,1]} < \infty$. Let Γ be a vector space generated by e_1 and e_2. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem 7.9, we have that μ-almost every continuous function f is nowhere differentiable.*

Interpretation 7.4 ([14], *propositions* 4.4 *and* 5.1) *Let f be a locally Lipschitz function defined on a non-empty open subset A of a separable Banach space $(B, ||\cdot||)$. Let $y \in B$ and $||y|| = 1$. Let $(e_k)_{k\in N}$ be a basis in B such that $e_1 = y$ and $\sum_{k\in\mathbb{N}} ||e_k|| < \infty$. Let Γ be a vector space generated by e_1. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem* 7.9*, we have*

$$\mu(\{x \in A : f'(x,y) \text{ does not exist}\}) = 0.$$

Interpretation 7.5 ([115], *Corollary* 3.2) *For $\varepsilon > 0$, the ε-square mesh for R^2 is defined as the collection of closed squares*

$\{[i\varepsilon,(i+1)\varepsilon] \times [j\varepsilon,(j+1)\varepsilon]\}_{i,j\in Z}$. *For totally bounded set $E \subset R^2$, define $N_\varepsilon(E) =$ of ε-mesh squares which meet E and*

$$\Delta(E) = \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(E)}{-\log \varepsilon}.$$

Let $I = [k2^{-m},(k+1)2^{-m}] \subset [0,1]$ be a dyadic interval, where $k,m \in N$ are fixed. For $f \in C[0,1]$, let $G_I(f) = \{(x, f(x))\}_{x\in I}$. Let $(e_k)_{k\in N}$ be a basis in $C[0,1]$ such that $e_1 = g,\ \Delta(G_I(g)) = 2$ (cf.[115]) and $\sum_{k\in\mathbb{N}} ||e_k||_{C[0,1]} < \infty$. Let Γ be a vector space generated by e_1. Then for the translation-invariant quasi-finite Borel measure μ constructed in Theorem 7.9*, we have*

$$\mu(\{f \in C[0,1] : \Delta(G_I(f)) \neq 2\}) = 0.$$

Remark 7.9 By using the technique applied in Appendix 15 .3(or 15.4), one can construct "Lebesgue measure"(or "Gaussian measure") on B and get similar interpretations (i.e., 7.1–7.5) in terms of such a measure in the system of axioms ZFC.

Finally, in the Solovay model, for an arbitrary infinite-dimensional separable Banach space B we have the following picture:

$$\mathbb{L}.\mathbb{N}.\mathbb{S}.(B) \subset \mathbb{L}.\mathbb{N}.\mathbb{S}.(B,\Gamma) \subset \mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B) \subseteq \mathbb{H}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{S}.\mathbb{S}.(B) \subseteq \mathbb{T}.\mathbb{N}.\mathbb{S}(B)$$

Remark 7.10 Let E be a non-empty set and let K be any σ-ideal of subsets of E. Let P be any sentence formulated for elements in E. We say that "almost every" element of E satisfies the property P with respect to the class K if a complement of the subset of E on which the property holds belongs to K. Clearly, if K_1 and K_2 are two σ-ideals of subsets of E such that $K_1 \subseteq K_2$, then the validity of the condition

(a) **"almost every " element of E satisfies the property P with respect to K_1**

implies the validity of the condition

(b) **"almost every " element of E satisfies the property P with respect to K_2.**

Using this argument and above-mentioned relation between σ-ideals of subsets $\mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}.(B)$ and $\mathbb{S}.\mathbb{S}.(B)$, we establish that propositions, obtained in [70](cf.pp.226-227), [14](cf.propositions 4.4 and 5.1) and [115](cf.Corollary 3.2) are not stronger than their interpretations 7.1-7.5 in Solovay's model.

In context of Theorems 7.5-7.9 we posed

Problem 7.1 Are the following differences

(i) $\mathbb{C}.\mathbb{N}.\mathbb{S}(B) \setminus \mathbb{L}.\mathbb{N}.\mathbb{S}.(B)$

(ii) $\mathbb{S}.\mathbb{S}(B) \setminus \mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}.(B)$

(iii) $\mathbb{Q}.\mathbb{F}.\mathbb{T}.\mathbb{N}.\mathbb{S}(B) \setminus \mathbb{L}.\mathbb{N}.\mathbb{S}.(B)$

(iv) $\mathbb{T}.\mathbb{N}.\mathbb{S}(B) \setminus \mathbb{S}.\mathbb{S}(B)$

non-empty in Solovay's model?

Below we demonstrate how relations can be changed between various classes of null sets in the case of non-separable Banach spaces. Let ℓ^∞ be a Banach space of all real-valued sequences equipped with norm $||\cdot||_\infty$ defined by

$$||(x_k)_{k\in N}||_\infty = \sup_{k\in N} |x_k|.$$

Lemma 7.4 *Let $r > 0$. Then in Solovay's model, there exists a translation-invariant Borel measure ν_r in ℓ^∞ such that every closed ball with radius r has ν_r-measure 1.*

Proof. We put

$B((x_k)_{k\in\mathbb{N}}) = (r(2x_k - 1))_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}} \in \mathbf{R}^{\mathbb{N}}$.

For the proof of Lemma 7.4 it suffices to put

$$(\forall X)(X \in B(l^\infty) \rightarrow \nu_r(X) = \mu_{\mathbb{N}}(B^{-1}(X))). \ \square$$

Remark 7.11 Lemma 7.4 gives a partial solution of one problem posed by D.H.Fremlin [45] and C.A.Rogers [152]: which Banach spaces have the property that there exists a translation-invariant Borel measure μ on U such that the closed unit ball has measure 1?

Theorem 7.10 *In Solovay's model an arbitrary closed ball in ℓ^∞ is a Haar null set in the sense of Christensen* [29].

Proof. Let X be a closed ball in ℓ^∞ with radius r. Let us denote by Y any closed ball with radius $r+1$. Applying the same argument as in the proof of Theorem 7.7 for the conditional probability Borel measure $\mu_{r+1}(\cdot|Y)$ and using the fact that an arbitrary Borel set is universally measurable, we easily establish the validity of Theorem 7.10. □

As ℓ^∞ is an union of countable family of non-degenerate closed balls and ℓ^∞ is not Haar null on itself, we obtain the validity of the following proposition.

Corollary 7.3 *In Solovay's model the class of Haar null sets in l^∞ is not σ-ideal.*

So an arbitrary bounded universally measurable set in ℓ^∞ can be put in any closed ball, using Theorem 7.11, we obviously obtain the validity of the following assertion.

Corollary 7.4 *In Solovay's model an arbitrary totally bounded by the norm universally measurable set in ℓ^∞ is a Haar null set in the sense of Christensen[29].*

Remark 7.12 Corollary 7.4 demonstrates that in Solovay model an arbitrary totally bounded closed subset is a Haar null set in ℓ^∞.

Combining Theorem 7.10 and the simple fact stating that every set with non-empty interior in an arbitrary topological vector space is not shy (cf.[70],p.222,Fact $2'$), we get

Corollary 7.5 $\mathbb{H}.\mathbb{N}.\mathbb{S}.(\ell^\infty) \setminus \mathbb{S}.\mathbb{S}.(\ell^\infty) \neq \emptyset$.

Remark 7.13 Corollary 7.5 shows that in Solovay's model analogous to Maxwell B.Stinchombe's result [168] stated that $\mathbb{H}.\mathbb{N}.\mathbb{S}.(B) = \mathbb{S}.\mathbb{S}.(B)$ for an arbitrary infinite-dimensional separable Banach space B is not valid for the non-separable Banach space ℓ^∞.

Chapter 8

On Essential Uniqueness

Let (E,S) be a measurable space. Let $\mathcal{K}$ be some class of σ-finite non-trivial measures defined on the measurable space (E,S). A measurable set $X \in S$ is said to have the property of essential uniqueness with respect to the class $\mathcal{K}$ if

$$(\forall Y)(\forall \mu)(\forall \lambda)(Y \in S \ \&\ \mu \in \mathcal{K} \ \&\ \lambda \in \mathcal{K} \rightarrow$$

$$\rightarrow \mu(X \cap Y) = \lambda(X \cap Y)).$$

The class of all measurable subsets of the space E whose every element has the property of essential uniqueness with respect to the class $\mathcal{K}$ is denoted by $S(\mathcal{K})$.

We begin our discussion with the following proposition.

Theorem 8.1 *The class $S(\mathcal{K})$ is a hereditary σ-ring (with respect to the σ-algebra S) of subsets of the space E, i.e.,*

$$(\forall X)(\forall Y)(X \in S(\mathcal{K}) \ \&\ Y \in S \ \&\ Y \subseteq X \rightarrow Y \in S(\mathcal{K})).$$

Proof. Let $X \in S(\mathcal{K})$. Then

$$(\forall Z)(\forall Y)(Z \in S \ \&\ Y \in S \rightarrow (\forall \mu)(\forall \lambda)(\mu \in \mathcal{K} \ \&\ \lambda \in \mathcal{K} \rightarrow$$

$$\rightarrow \mu(X \cap (Z \cap Y)) = \lambda(X \cap (Z \cap Y)))).$$

Note that if $Y \subseteq X$, then

$$\mu(X \cap (Z \cap Y)) = \mu(Y \cap Z),$$

$$\lambda(X \cap (Z \cap Y)) = \lambda(Y \cap Z).$$

In view of the relation considered above, we have

$$(\forall Z)(Z \in S \rightarrow (\forall \mu)(\forall \lambda)(\mu \in S(\mathcal{K}) \ \&\ \lambda \in S(\mathcal{K}) \rightarrow$$

$$\rightarrow \mu(Y \cap Z) = \lambda(Y \cap Z))).$$

This proves the validity of the relation $Y \in S(\mathcal{K})$.

Let us show that

$$(\forall X)(\forall Y)(X \in S(\mathcal{K}) \ \&\ Y \in S(\mathcal{K}) \to X \cup Y \in S(\mathcal{K})).$$

Indeed, since

$$(\forall Z)(Z \in S \to (\forall \mu)(\forall \lambda)(\mu \in \mathcal{K} \ \&\ \lambda \in \mathcal{K} \to$$

$$\to \mu(Z \cap (X \setminus (X \cap Y))) = \lambda(Z \cap (X \setminus (X \cap Y))) \ \&$$

$$\mu(Z \cap (X \cap Y)) = \lambda(Z \cap (X \cap Y))) \ \&\ \mu(Z \cap (Y \setminus (X \cap Y))) = \lambda(Z \cap (Y \setminus (X \cap Y)))),$$

we have

$$\mu(Z \cap (X \cup Y)) = \lambda(Z \cap (X \cup Y)).$$

The latter relation implies

$$X \cup Y \in S(\mathcal{K}).$$

Let $(X_k)_{k \in N}$ be an arbitrary sequence of elements of the class $S(\mathcal{K})$. Using the latter relation, we can assume that the family $(X_k)_{k \in N}$ is disjoint.

Note that

$$(\forall k)(\forall Z)(k \in N \ \&\ Z \in S(\mathcal{K}) \to (\forall \mu)(\forall \lambda)(\mu \in S(\mathcal{K}) \ \&$$

$$\&\ \lambda \in S(\mathcal{K}) \to \mu(Z \cap X_k) = \lambda(Z \cap X_k)).$$

Using the property of σ-finiteness of measures μ and λ, we can assume

$$(\forall k)(k \in N \to 0 < \mu(X_k) < \infty \ \&\ 0 < \lambda(X_k) < \infty).$$

We have

$$\mu(Z \cap (\bigcup_{k \in N} X_k)) = \sum_{k \in N} \mu(X_k \cap Z) =$$

$$= \sum_{k \in N} \lambda(X_k \cap Z) = \lambda(Z \cap (\bigcup_{k \in N} X_k)),$$

which means the validity of the relation

$$\bigcup_{k \in N} X_k \in S(\mathcal{K}).$$

The proof is completed. $\square$

Remark 8.1 One can easily construct an example of the measurable space (E, S) with a class $\mathcal{K}$ of σ-finite nontrivial measures such that $S(\mathcal{K})$ is not a σ-algebra.

The following theorem concerns the existence of a maximal (in the sense of measure) element from the class $S(\mathcal{K})$.

Theorem 8.2 *There exists an element $X_0 \in S(\mathcal{K})$ such that*

$$(\forall Z)(\forall \mu)(Z \in S(\mathcal{K}) \ \&\ \mu \in \mathcal{K} \to \mu(Z \setminus X_0) = 0).$$

The set X_0 is a maximal (in the sense of measure) subset of the space E which belongs to the class $S(\mathcal{K})$.

Proof. Let λ be an arbitrary element of the class $\mathcal{K}$. Consider a probability measure $\widetilde{\lambda}$ which is equivalent to the measure λ.

Denote
$$a = \sup_{X \in S(\mathcal{K})} \widetilde{\lambda}(X).$$

It is clear that
$$(\forall n)(n \in N \to (\exists X_n)(X_n \in S(\mathcal{K}) \;\&\; \widetilde{\lambda}(X_n) > a - \frac{1}{n})).$$

Let us consider the set
$$X_0 = \bigcup_{k \in N} X_k.$$

Using Theorem 8.1, we have
$$X_0 \in S(\mathcal{K}).$$

Let $Z \in S(\mathcal{K}),\ \mu \in \mathcal{K}$ and $\mu(Z \setminus X_0) > 0$.

It is clear that $\alpha = \widetilde{\lambda}(Z \setminus X_0)$ is also a strictly positive number. Note that
$$Z \setminus X_0 \in S(\mathcal{K}) \textit{ and } \lambda|_{Z \setminus X_0} = \mu|_{Z \setminus X_0}.$$

Therefore, we have
$$\widetilde{\lambda}((Z \setminus X_0) \cup X_0) = a + \alpha > a,$$
and we get a contradiction with the definition of the number a.

Theorem 8.2 is proved. □

Below we give one construction closely connected with Theorem 8.2.

Let Γ be a subgroup of the group $\mathbf{R}^N$ such that:

1) If $(g_1, g_2, \cdots) \in \Gamma$, then $(\forall n)(n \in N \to (g_1, \cdots, g_n, 0, 0, \cdots) \in \Gamma)$,

2) $\Gamma \subseteq G_{[0;1]^N}$, where $G_{[0;1]^N}$ is a subgroup of the group $\mathbf{R}^N$ considered in Chapter 5.

Let us denote the measure $\nu_{[0;1]^N}$ by ν (see Chapter 5).

Assume also that
$$(\forall n)(n \in N \to A_n = \prod_{i=1}^{n} \mathbf{R}_i \times \prod_{i>n} \triangle_i),$$
where
$$(\forall i)(i \in N \to \mathbf{R}_i = \mathbf{R} \;\&\; \triangle_i = [0;1]).$$

The following theorem is valid.

Theorem 8.3 *The set $\bigcup_{n \in N} A_n$ has the property of essential uniqueness with respect to the class $\mathcal{K}_0$ of all Γ-invariant σ-finite Borel measures, taking the value one on the element $[0;1]^N$ if and only if the group Γ is everywhere dense in the space $\mathbf{R}^N$ with respect to the usual Tykhonoff topology.*

Proof. Assume that $\bigcup_{n \in N} A_n$ has the property of essential uniqueness with respect to the class $\mathcal{K}_0$ of all Γ-invariant σ-finite Borel measures taking the value one on the element $[0;1]^N$, i.e.,
$$(\forall X)(\forall \mu)(\forall \lambda)(X \in \mathcal{B}(\mathbf{R}^N) \;\&\; \mu \in \mathcal{K}_0 \;\&\; \lambda \in \mathcal{K}_0 \to$$

$$\to \mu(X \cap (\bigcup_{n\in N} A_n)) = \lambda(X \cap (\bigcup_{n\in N} A_n))).$$

Denote by $B^{(n)}_{x,r}$ the open ball with the center at a point x and radius r in $\mathbf{R}^n$, i.e.,

$$B^{(n)}_{x,r} = \Big\{ y | y \in \mathbf{R}^N \ \& \ \sum_{i=1}^{n} (y_i - x_i)^2 < r^2 \Big\},$$

where $n \in N,\ x = (x_1, \cdots, x_n) \in \mathbf{R}^N$.

Assume that the group Γ is not everywhere dense in $\mathbf{R}^N$. Then, for some natural number $n \in N$, there exists a real positive number $r > 0$ and a point $x \in \mathbf{R}^n$ such that

$$(B^{(n)}_{x,r} \times \mathbf{R}^{N \setminus \{1,\cdots,n\}}) \cap \Gamma = \emptyset.$$

Let us consider the set $B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i$.

It is clear that

$$\nu(B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i) = b_n(B^{(n)}_{0,\frac{r}{4}}) > 0,$$

where by b_n is denoted the n-dimensional standard Borel measure in $\mathbf{R}^n$.

Since the measure ν is σ-finite, there exists a countable Γ-configuration B of the set $B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i$ which is ν-almost Γ-invariant.

Note that the condition

$$B^{(n)}_{x,\frac{r}{4}} \times \prod_{i>n} \triangle_i \subseteq \mathbf{R}^N \setminus B$$

is valid.

Indeed, assume the contrary. Then, for some element $g \in \Gamma$, we have

$$g(B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i) \cap (B^{(n)}_{x,\frac{r}{4}} \times \prod_{i>n} \triangle_i) \neq \emptyset.$$

Hence, there exists a point $\widetilde{x} = (x_1, \cdots) \in B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i$ such that the condition

$$g(\widetilde{x}) \in B^{(n)}_{x,\frac{r}{4}} \times \prod_{i>n} \triangle_i$$

is valid, where

$$g(\widetilde{x}) = (g_1 + x_1, g_2 + x_2, \cdots, g_n + x_n, \cdots).$$

By the property of the group Γ, we have

$$\widetilde{g} = (g_1, \cdots, g_n, 0, 0, \cdots) \in \Gamma.$$

Also, for $\widetilde{g} \in \Gamma$, we have

$$\widetilde{g}(B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i) \cap (B^{(n)}_{x,\frac{r}{4}} \times \prod_{i>n} \triangle_i) \neq \emptyset,$$

from which we get

$$\widetilde{g}(B^{(n)}_{0,\frac{r}{4}} \times \prod_{i>n} \triangle_i) \subseteq B^{(n)}_{x,r} \times \prod_{i>n} \triangle_i,$$

i. e.,

$$\widetilde{g} = \widetilde{g}(0,0,\cdots) = (g_1,\cdots,g_n,0,0,\cdots) \in B_{x,r}^{(n)} \times \mathbf{R}^{N\setminus\{1,\cdots,n\}}$$

which is a contradiction.

The necessity is proved.

Let us prove the sufficiency.

Let μ be an arbitrary element of $\mathcal{K}_0$.

Since Γ is everywhere dense in $\mathbf{R}^N$, the set of all elements $(g_1,g_2,\cdots,g_{n+k}) \in \mathbf{R}^{n+k}$, where

$$(g_1,\cdots,g_{n+k},0,0,\cdots) \in \Gamma,$$

is everywhere dense in $\mathbf{R}^{n+k}$.

By the property of essential uniqueness of the standard Borel measure b_{n+k} in the space $\mathbf{R}^{n+k}$, we have

$$(\forall X)(X \in \mathcal{B}(\mathbf{R}^{n+k}) \to \mu(X \times \prod_{i>n+k} \triangle_i) = b_{n+k}(X)).$$

Thus, we have obtained

$$\nu(X \times \prod_{i>n+k} \triangle_i) = \mu(X \times \prod_{i>n+k} \triangle_i).$$

Using the Carathéodory theorem, we easily conclude that

$$(\forall Y)(Y \in B(\mathbf{R}^N) \to \mu(Y \cap A_n) = \nu(Y \cap A_n)).$$

The latter relation means that a subset A_n has the property of essential uniqueness with respect to the class $\mathcal{K}_0$ for arbitrary $n \in N$.

An application of Theorem 8.1 completes the proof of the sufficiency. $\square$

Let us consider the following application of Theorem 8.3.

Theorem 8.4 *The set $\bigcup_{n\in N} A_n$ is a maximal (in the sense of measure) subset of the space $\mathbf{R}^N$ having the property of essential uniqueness in the class $\mathcal{K}_0$, i.e.,*

$$(\forall X)(\forall \mu)(X \in S(\mathcal{K}_0) \ \& \ \mu \in \mathcal{K}_0 \to \mu(X \setminus \bigcup_{n\in N} A_n) = 0).$$

Proof. By Theorem 8.3, we can easily prove that

$$(\forall X)(\forall \mu)(X \in S(\mathcal{K}_0) \ \& \ \mu \in \mathcal{K}_0 \to \mu(X \setminus \bigcup_{n\in N} A_n) = 0).$$

Assume the contrary and let, for some set $Z_0 \in S(\mathcal{K}_0)$ and some measure $\mu_0 \in \mathcal{K}_0$,

$$\mu_0(Z_0 \setminus \bigcup_{n\in N} A_n) > 0.$$

It is clear that $Z_0 \setminus \bigcup_{n\in N} A_n \in S(\mathcal{K}_0)$.

On the other hand, for the measure ν, we have

$$\nu(Z_0 \setminus \bigcup_{n\in N} A_n) = 0,$$

which is a contradiction with the condition $Z_0 \in S(\mathcal{K}_0)$ since

$$\mu_0(Z_0 \setminus \bigcup_{n\in N} A_n) \neq \nu(Z_0 \setminus \bigcup_{n\in N} A_n). \quad \square$$

Example 8.1 Let (E,S) be a measurable space, $\mathcal{K}$ be some class of σ-finite measures defined on the space (E,S). We say that a set $X \subseteq E$ has the property of being single-valued in the class $\mathcal{K}$ if

$$(\forall \mu)(\forall \lambda)(\mu \in \mathcal{K} \,\&\, \lambda \in \mathcal{K} \rightarrow \mu(X) = \lambda(X)).$$

It is clear that every element of the class $S(\mathcal{K})$ has the property of being single-valued.

On the other hand, we can easily construct an example of the measurable space (E,S) with a class of σ-finite nontrivial measures, for which the class of all subsets with the property of being single-valued is larger than the class $S(\mathcal{K})$.

Example 8.2 Let $\Gamma \subseteq \mathbf{R}^N$ be a subgroup of $\mathbf{R}^N$ such that

$$(\forall g)(g = (g_1,\cdots,g_n,\cdots) \,\&\, g \in \Gamma \rightarrow (\forall n)(n \in N \rightarrow$$

$$\rightarrow (g_1,\cdots,g_n,0,0,\cdots) \in \Gamma)).$$

Note that the set $\bigcup_{n\in N} A_n$ has the property of essential uniqueness in the class of all σ-finite Γ-invariant Borel measures taking value 1 on the element $[0;1]^N$ if and only if the following conditions are satisfied:

1) card$(\Gamma/G_{[0;1]^N}) \leq \aleph_0$;

2) The group Γ is everywhere dense in the space $\mathbf{R}^N$ with respect to the Tykhonoff topology.

Example 8.3 If the conditions mentioned in Example 8.2 are satisfied, then:

a) For $2 \leq \text{card}(\Gamma/G_{[0;1]^N}) \leq \aleph_0$, the element $\bigcup_{n\in N} A_n$ is not a maximal (in the sense of measure) subset of the space $\mathbf{R}^N$ with the property of essential uniqueness in the class $\mathcal{K}_0$.

b) For $2 \leq card(\Gamma/G_{[0;1]^N}) \leq \aleph_0$, we can easily construct an example of a maximal (in the sense of measure) subset of the space $\mathbf{R}^N$ with the property of essential uniqueness in the class of all Γ-invariant σ-finite Borel measures taking the value one on the element $[0;1]^N$.

Let (E,S,G) be a G-invariant measurable space and P be a some sentence given in terms of measure. Let us denote by $K(P)$ the class of all G-invariant σ-finite measures defined on (E,S,G) satisfying P. We say that a measure μ has the property of uniqueness in the class $K(P)$ if for any nontrivial $\lambda \in K(P)$ the following equality $\lambda = \mu$ holds.

We remind the reader that a G-invariant measure μ has a metrically transitivity property if

$$(\forall X)(X \in S \,\&\, \mu(X) > 0 \rightarrow (\exists (g_k)_{k\in N})(\forall k)(k \in N \rightarrow g_k \in G) \,\&$$

$$(\mu(E \setminus \cup_{k\in N} g_k(X)) = 0)).$$

We say that a transformation $g : E \to E$ acts freely if

$$(\forall x_1)(\forall x_2)(x_1 \in E \ \& \ x_2 \in E \ \& \ x_1 \neq x_2 \to g(x_1) \neq g(x_2)).$$

The following classical result is due to A.B. Kharazishvili (see, e.g.[85]).

Lemma 8.1 *Let (E,G,S) be a $G-$invariant measurable space, i.e., the following two conditions*
(i) G is a group of transformations of E;
(ii) S is a G-invariant σ-algebra of subsets of E,
hold.
Let μ be a σ-finite G-invariant measure on (E,G,S) and let G consist of a freely acting uncountable subgroup. Then for any $X \in S$ with $\mu(X) > 0$ there exists a μ-nonmeasurable part of X.

Proof. Without loss of generality, we can assume $0 < \mu(X) < \infty$. Let us denote by G_1 a freely acting subgroup of G having cardinality $\aleph_1$. Let $G_1 = (g_\xi)_{\xi<\omega_1}$. Let us consider partition of E into the intransitive classes of the group G_1. Let us denote by $(K_i)_{i\in I}$ injective family of all intransitive classes having nonempty intersection with the set X. Let Y be a system of elements from $(K_i \cap X)_{i\in I}$ such that

$$(\forall i)(i \in I \to \mathrm{card}(Y \cap (K_i \cap X)) = 1).$$

It is clear that
(i) $X \subset \cup_{g\in G_1} g(Y)$;
(ii) $(\forall f)(\forall g)(f \in G_1 \ \& \ g \in G_1 \ \& \ f \neq g \to f(Y) \cap g(Y) = \emptyset)$.
We set $X_\xi = X \cap g_\xi(Y)$ for $\xi < \omega_1$. Clearly $(X_\xi)_{\xi<\omega_1}$ is disjoint and its union covers X. Let us show that we can indicate $J_0 \subseteq [0;\omega_1[$, for which the union $\cup_{\xi\in J_0} X_\xi$ is μ-nonmeasurable. Indeed, let us assume the contrary and let for any $J \subseteq [0;\omega_1[$ a set $\cup_{\xi\in J} X_\xi$ be μ-measurable. Then a functional λ, given by

$$(\forall J)(J \subseteq [0;\omega_1[\to \lambda(J) = \mu(\cup_{\xi\in J} X_\xi)),$$

is a finite measure defined on the class of all subsets of $[0;\omega_1[$ such that $\lambda(\{\xi\}) = 0$ for any $\xi < \omega_1$. This fact is a simple consequence of the following ones:
(i) $(g(X_\xi))_{\xi\in G_1}$ is an uncountable family of disjoint μ-measurable sets;
(ii) the measure μ, being σ-finite, has a Suslin property (i.e., cardinality of any family of disjoint μ-measurable sets with positive μ-measure is countable).
Hence, we deduce that the cardinal number $\aleph_1$ is measurable, which contradicts the well-known result of Ulam about nonmeasurability of the cardinal number $\aleph_1$(in this context see e.g., [86]).
Let J_0 be a subset of $[0;\omega_1[$ such that $\cup_{\xi\in J_0} X_\xi$ is μ-nonmeasurable. As $\cup_{\xi\in J_0} X_\xi \subset X$, the proof of Lemma 8.1. is complete. □

Theorem 8.5 (A.B.Kharazishvili) *Let E be a basic space, G be a group of transformations of E containing freely acting uncountable subgroup, λ be a complete σ-finite G-invariant measure with the property of metrically transitivity. Then for any G-invariant measure μ with $dom(\mu) = dom(\nu)$ there exists $q > 0$ such that $\lambda = q\mu$.*

Proof. Let μ be an arbitrary σ-finite G-invariant measure defined on $\mathrm{dom}(\lambda)$. Let us show that the measure μ is absolutely continuous with respect to λ. Assume the contrary and suppose that there is $X \in \mathrm{dom}(\lambda)$ with $\lambda(X) = 0$ and $\mu(X) > 0$. By Lemma 8.1 we can indicate a μ-nonmeasurable subset of X contradicting the completeness of the measure λ.

By the well-known Radon-Nikodym theorem (see,e.g.,[54]) there exists λ-measurable function $f : E \to \mathbf{R}^+$ such that

$$(\forall Z)(Z \in \mathrm{dom}(\nu) \to \mu(Z) = \int_Z f d\lambda).$$

Let g be an element of G. Then for $Z \in \mathrm{dom}(\nu)$

$$\int_Z (f \circ g) d\lambda = \int_{g(Z)} f d\lambda = \mu(g(Z)) = \mu(Z) = \int_Z f d\lambda.$$

Since above-mentioned relation holds for any $Z \in \mathrm{dom}(\nu)$, we have $(f \circ g)(x) = f(x)$ for λ-almost every point in E.

For any positive numbers a and b we set

$$Z_{a,b} = \{x : x \in E, a \leq f(x) \leq b\}.$$

Hence, the condition $\lambda(g(Z_{a,b}) \triangle Z_{a,b}) = 0$ is valid for an arbitrary $g \in G$. As the group G consists of uncountable freely acting transformations of E, we conclude that

$$\lambda(E \setminus Z_{a,b}) = 0 \bigvee \lambda(Z_{a,b}) = 0.$$

This allows us to construct a countable family of intervals $([a_m; b_m])_{m \in N}$ such that
(i) $(\forall m)(m \in N \to [a_{m+1}, b_{m+1}] \subset [a_m, b_m])$;
(ii) $\lim_{m \to \infty}(b_m - a_m) = 0$;
(iii) $(\forall m)(m \in N \to \lambda(E \setminus Z_{a_m, b_m}) = 0)$.

Let q be a common point of above-mentioned segments, then it is clear that the function f coincides with q λ-almost everywhere in E. Thus Theorem 8.5 is completely proved. $\square$

Remark 8.2 The proof of Theorem 8.5 can be found also in [85].

Theorem 8.6 *Let ν_Δ be a Borel measure constructed in Lemma 5.1 for $\Delta = [0,1[^N$. Let ν be a completion of the measure ν_Δ. Then the measure ν has the property of uniqueness in the class of all σ-finite G-invariant measures $(\mathbf{R}^{(N)} \subseteq G \subseteq \ell_1)$ defined on dom(ν) and obtaining a numerical value one on $[0;1[^N$.*

Proof. Following Theorem 8.5, we have to show that the measure ν has a metrically transitivity property since the group $\mathbf{R}^{(N)}$ is a freely acting uncountable group of transformations of $\mathbf{R}^N$. Note that it is enough to show that the measure ν_Δ has an analogous property. Assume the contrary and let ν_Δ have no metrically transitivity property. Then there exists $B \in B(\mathbf{R}^N)$ with $0 < \nu_\Delta(B) < \infty$ such that for any family of transformations $(g_k)_{k \in N}$, where $g_k \in \mathbf{R}^{(N)}$,

$$\nu_\Delta(\mathbf{R}^N \setminus \cup_{k \in N} g_k(B)) > 0.$$

From the σ-finiteness of the measure ν_Δ we establish the existence of a sequence $(g_k^{(0)})_{k\in N}$ of transformations of $\mathbf{R}^{(N)}$ such that

$$(\forall g)(g\in\mathbf{R}^{(N)}\to\nu_\Delta(\cup_{k\in N}g_k^{(0)}(B)\triangle g(\cup_{k\in N}g_k^{(0)}(B)))=0).$$

Note that $\nu_\Delta([0;1[^N\setminus\cup_{k\in N}g_k^{(0)}(B))>0$. Indeed, if $\nu_\Delta([0;1[^N\setminus\cup_{k\in N}g_k^{(0)}(B))=0$, then

$$\nu_\Delta(\cup_{n\in N}A_n\setminus\cup_{g\in Z^{(N)}}g(\cup_{k\in N}g_k^{(0)}(B)))=0,$$

which contradicts the condition

$$\nu_\Delta(\mathbf{R}^N\setminus\cup_{k\in N}g_k(B))>0$$

being valid for any sequence $(g_k)_{k\in N}$ of elements $\mathbf{R}^{(N)}$.

Let $X=\cup_{k\in N}g_k^{(0)}(B)$ and $Y=\mathbf{R}^N\setminus X$.

Let us define two $\mathbf{R}^{(N)}$- invariant orthogonal σ-finite Borel measures μ_1 and μ_2 by

$$(\forall Z)(Z\in B(\mathbf{R}^N)\to\mu_1(Z)=\frac{1}{\nu_\Delta(X\cap[0;1[^N)}\nu_\Delta(Z\cap X)\ \&$$

$$\mu_2(Z)=\frac{1}{\nu_\Delta(Y\cap[0;1[^N)}\nu_\Delta(Z\cap Y)).$$

Obviously, both measures obtain the numerical value one on $[0;1[^N$.

For $W\in B(\mathbf{R}^n)(n\in N)$ we set

$$\lambda_1(W)=\mu_1(W\times[0;1[^{N\setminus\{1,\cdots,n\}}),$$

$$\lambda_2(W)=\mu_2(W\times[0;1[^{N\setminus\{1,\cdots,n\}}).$$

Since λ_1 and λ_2 are $\mathbf{R}^n$ -invariant Borel measures in $\mathbf{R}^n$ with the numerical value one on $[0;1[^n$, using the well-known Sierpinski result, we conclude that $\lambda_1=\lambda_2=b_n$. Hence,

$$\mu_1(W\times[0;1[^{N\setminus\{1,\cdots,n\}})=\mu_2(W\times[0;1[^{N\setminus\{1,\cdots,n\}}).$$

Since the class

$$\{W\times[0;1[^{N\setminus\{1,\cdots,n\}}: n\in N\ \&\ W\in B(\mathbf{R}^n)\}$$

generate the σ-algebra $B(\mathbf{R}^N)\cap(\cup_{n\in N}A_n)$, by the Caratheodory theorem (see, e.g., [54]) we obtain

$$(\forall Z)(Z\in B(\mathbf{R}^N)\cap(\cup_{n\in N}A_n)\to\mu_1(Z)=\mu_2(Z)).$$

Since

$$\mu_1(\mathbf{R}^N\setminus\cup_{n\in N}A_n)=\mu_2(\mathbf{R}^N\setminus\cup_{n\in N}A_n)=0,$$

we have

$$(\forall Z)(Z\in B(\mathbf{R}^N)\to\mu_1(Z)=\mu_2(Z)).$$

The last relation contradicts the fact that two measures μ_1 and μ_2 are orthogonal, and the metrically transitivity of the measure ν_Δ is proved. Since ν is the completion of the measure ν_Δ, it has the property of metrical transitivity.

Setting $E = \mathbf{R}^N,\ G = \mathbf{R}^{(N)},\ \nu = \mu$ and applying Theorem 8.5, we establish that for any G-invariant measure μ defined on $\mathrm{dom}(\nu)$ with the numerical value one on $[0;1[^N$ there exists $q > 0$ such that $\nu = q\mu$. For the set $[0;1[^N$ we get the validity of the equality

$$\nu([0;1[^N) = q\mu([0;1[^N),$$

which implies $q = 1$. Hence, $\mu = \nu$ and Theorem 8.6 is proved. $\square$

Now we consider some applications of Theorem 8.5.

Let T be a measurable transformation on the measure space (X,S,ν) such that νT^{-1} is absolutely continuous with respect to ν. Then there exists a non-negative ν-measurable function Φ on X such that

$$\nu(T^{-1}(X)) = \int_X \Phi(y)d\nu(y)$$

for every ν-measurable subset of X.

The function Φ plays the role of the Jacobian $J(T^{-1})$ of the transformation T^{-1}(or, rather the absolute value of the Jacobian)(see, e.g., [54]) in the theory of transformations of multiple integrals. It is clear that $J(T^{-1})$ coincides with a Radon-Nikodym derivative $\frac{d\nu T^{-1}}{d\nu}$, which is unique a.e. with respect to ν(some properties of Radon-Nikodym derivatives (i.e. Jacobians) can be found also in the above-mentioned book).

Let us consider n-dimensional Euclidean space $\mathbf{R}^n$ and n-dimensional standard Lebesgue measure μ_n. Let T be a linear μ_n-measurable transformation of E_n. Then

$$\frac{d\mu_n T^{-1}}{d\mu_n}(x) = \lim_{k\to\infty} \frac{\mu_n(T^{-1}(U_k(x)))}{\mu_n(U_k(x))},$$

where $U_k(x)$ is a spherical neighborhood with the center in x and radius $\mathbf{r}_k > 0$ so that $\lim_{k\to\infty} r_k = 0$.

It is reasonable to note, that in a particular situation when

$$(\forall h)(h \in \mathbf{R}^n \to T^{-1}(h) \neq \emptyset),$$

we have the following different simple method of calculation of Radon-Nikodym derivative(i.e., Jacobian of T^{-1} with respect to measure μ_n): firstly note that the measure μ_n satisfies all conditions of Theorem 8.5. Let us show that the measure $\mu_n T^{-1}$ is a translation-invariant measure defined on $\mathrm{dom}(\mu_n)$. Indeed, by the definition of the measure $\mu_n T^{-1}$ we have

$$(\forall X)(X \in \mathrm{dom}(\mu_n) \to \mu_n T^{-1}(X) = \mu_n(T^{-1}(X))).$$

Let $h \in \mathbf{R}^n$ and h_0 be an element of $\mathbf{R}^n$ such that $T(h_0) = h$, then

$$(\forall X)(X \in \mathrm{dom}(\mu_n) \to T^{-1}(X+h) = T^{-1}(X) + h_0).$$

Hence,

$$(\forall X)(\forall h)(X \in \mathrm{dom}(\mu_n)\ \&\ h \in \mathbf{R}^n \to \mu_n T^{-1}(X+h) =$$

$$= \mu_n(T^{-1}(X+h)) = \mu_n(T^{-1}(X)+h_0) = \mu_n(T^{-1}(X)) = \mu_n T^{-1}(X)),$$

which means a translation-invariance of the measure $\mu_n T^{-1}$.

By Theorem 8.5, we conclude that there exists a constant $q > 0$ such that

$$(\forall X)(X \in \mathrm{dom}(\mu_n) \to \mu_n(T^{-1}(X)) = q\mu_n(X)).$$

Setting $X = [0;1]^n$, we obtain $q = \mu_n(T^{-1}([0;1]^n))$.
Finally, we get

$$\frac{d(\mu_n T^{-1})}{d\mu_n}(x) = \mu_n(T^{-1}([0;1]^n))$$

a.e. with respect μ_n.

Remark 8.3 According to Theorem 8.5, we have

$$(\forall X)(X \in B(\mathbf{R}^n) \to \mu_n T^{-1}(X) = \mu_n(T^{-1}([0;1]^n))\mu_n(X)).$$

If we consider analogous problem for measure ν constructed in Lemma 5.1, we observe that an application of the first method is not possible since the ν-measure of any open set in $\mathbf{R}^N$ is equal to $+\infty$, whenever the second method works out successfully. In particular, the following result is valid.

Theorem 8.7 *Let $T : \mathbf{R}^N \to \mathbf{R}^N$ be a linear ν-measurable mapping such that for some G ($\mathbf{R}^{(N)} \subseteq G \subseteq \ell_1$) the following relation*

$$(\forall h)(h \in G \to T^{-1}(h) \cap \ell_1 \neq \emptyset)$$

is valid. Then $\frac{d\nu T^{-1}}{d\nu}(x) = \nu(T^{-1}([0;1[^N))$ a.e. with respect to the measure ν, i.e., the number $\nu(T^{-1}([0;1[^N))$ coincides with the Jacobian of the mapping T^{-1}.

Let M be a class of all linear operators as in Theorem 8.7.

Example 8.4 Let T_A be a linear operator in $\mathbf{R}^N$ generated by any element $A \in G$(cf. Theorem 3.19). Then $T_A \in M$ for $A \in G$ and $J(T_A) = 1$.

Chapter 9

On the Erdös-Sierpiński Duality Principle

We begin our discussion of some properties of subsets of the second category and of non-trivial σ-finite Borel measures in infinite-dimensional Polish topological vector spaces and the study duality between such measures and the Baire category. The reader can see main notions and facts which we apply below e.g. in [54],[86],[124].

Let (E,T) be a topological vector space. Denote by $B(E)$ the Borel σ-algebra of subsets of the space E. Consider a nontrivial Borel measure μ defined on the σ-algebra $B(E)$. A subset $X \subseteq E$ is called small in the sense of measure if $\mu^*(X) = 0$. Analogously, a subset $Y \subseteq E$ is called small in the sense of category if it is a first category set in E. Further, let P be a sentence formulated only by using the notions of measure zero, of the first category and of purely set-theoretical notions. We say that the duality principle between the measure μ and the Baire category is valid with respect to the sentence P if the sentence P is equivalent to the sentence P^* obtained from the sentence P by interchanging in it the notions of the above small sets. We also say that the strict duality principle between the measure μ and Baire category is valid if the duality between the measure μ and the Baire category is valid for all P sentences formulated only by using the notions of measure zero of the first category and of purely set-theoretical notions.

One important well-known property of second category subsets in infinite-dimensional Polish topological vector spaces is presented in the following

Lemma 9.1 *Let E be an arbitrary infinite-dimensional Polish topological vector space. Then*
$((\forall X)(X \subseteq E$ & X *is a Baire subset of second category* $\rightarrow$
there exists a neighborhood V *of the zero vector* $\mathbf{0}$ $)$&
$(\forall h)(h \in V \rightarrow X \cap (X+h)$ *is a second category set*$))$.

Proof. Since the set X has the Baire property, there exists an open subset $G \subseteq E$ and a first category subset $P \subseteq E$ such that the equality

$$X = G \triangle P$$

is fulfilled.

Evidently, there exists an open nonempty neighborhood U of the zero vector $\mathbf{0}$ such that $-U = U$ and for some $h \in E$ the equality $U^* + h \subseteq G$ holds where $U^* = U - U$.

Note that the inclusion

$$[(U^* + x + h) \cap (U^* + h)] \setminus [P \cup (P + x)] \subseteq (X + x) \cap X$$

holds for arbitrary $x \in E$. If we consider $x \in U$, then

$$[(U + x + h) \cap (U + h)] \setminus [P \cup (P + x)] \subseteq$$

$$\subseteq [(U^* + x + h) \cap (U^* + h)] \setminus [P \cup (P + x)] \subseteq (X + x) \cap X.$$

Using the well known Baire theorem (see, e.g.,[86]) we conclude that the set

$$[(U + h + x) \cap (U + h)] \setminus [P \cup (P + h)] = [[(U + x) \cap U] + h] \setminus [P \cup (P + x)]$$

is not empty, because it coincides with a h-translation of the nonempty open set $(U + x) \cap U$ minus a first category set $[P \cup (P + x)]$.

The proof of Lemma 9.1 is completed. $\square$

Remark 9.1 The method considered in the proof of Lemma 9.1 has been worked out and applied by many authors, e.g., by J.Oxtoby who established an analogous result in [124] for linear Baire second category subsets in R.

The next theorem plays the main role in our further consideration.

Lemma 9.2 *Let E be an infinite-dimensional Polish topological vector space. Then for an arbitrary countable family of compact subsets $(K_i)_{i \in N}$ in E and for an arbitrary neighborhood V of the zero vector $\mathbf{0}$ there exists an element $h_0 \in V$ such that*

$$(\bigcup_{i \in N} K_i + h_0) \bigcap (\bigcup_{i \in N} K_i) = \emptyset.$$

Proof. Note that E can be considered as a topological group under the usual addition operation "+". Let us consider an union

$$\bigcup_{(i,j) \in N \times N} (K_i - K_j).$$

For arbitrary indices $i \in N$ and $j \in N$ the set $K_i - K_j$ is compact because it is an image of the compact set $K_i \times K_j$ under the continuous mapping

$$\varphi : (x, y) \to x - y \ \ ((x, y) \in K_i \times K_j).$$

Since the space E is a Baire space and an arbitrary compact subset of E is nowhere dense in it, we conclude that the union

$$\bigcup_{(i,j) \in N \times N} (K_i - K_j)$$

is the first category subset in E.

Let us consider an arbitrary neighborhood V of the zero vector $\mathbf{0}$. Note that

$$V \setminus \bigcup_{(i,j)\in N\times N} (K_i - K_j) \neq \emptyset,$$

because V is the set of the second category in E. Hence there exists an element h_0 such that

$$h_0 \in V \setminus \bigcup_{(i,j)\in N\times N} (K_i - K_j).$$

One can easily check the validity of an equality

$$(\bigcup_{i\in N} K_i + h_0) \bigcap (\bigcup_{i\in N} K_i) = \emptyset.$$

Lemma 9.2 is proved. $\square$

Analogously, we can get the validity of the following assertion

Lemma 9.3 *Let $\mathbf{R}^{\omega}$ be a vector space of all real sequences equipped with Tykhonoff topology. Then for an arbitrary countable family of compact subsets $(K_i)_{i\in N}$ in $\mathbf{R}^{\omega}$ and for an arbitrary neighborhood V of the zero vector $\mathbf{0}$ there exists an element $h_0 \in V$ such that*

$$(\bigcup_{i\in N} K_i + h_0) \bigcap (\bigcup_{i\in N} K_i) = \emptyset.$$

Let us denote by (S) the following statement:

$$(\forall X)(X \subseteq E \ \& \ X \text{ is a Baire subset of second category } \rightarrow$$

$$(\forall \varepsilon)(\varepsilon > 0 \rightarrow (\text{there exists a neighborhood } V_{\varepsilon} \text{ of the zero vector } \mathbf{0})\&$$

$$(\forall h)(h \in V_{\varepsilon} \rightarrow X \cap (X+h) \text{ is a second category set}))).$$

Remark 9.2 By Remark 9.1 and the Steinhaus property of the linear Lebesgue measure it is easy to obtain the validity of the duality between the linear Lebesgue measure and the Baire category with respect to the sentence (S) in R. This result is essentially due to Oxtoby[124] and may be called the Oxtoby duality principle in R.

Corollary 9.1 Using Lemma 9.1 and Theorem 9.1, we conclude that the Oxtoby duality principle between an arbitrary nontrivial σ-finite Borel measure and the Baire category is not valid in the infinite-dimensional Polish topological vector Spaces. An analogous result is valid also in the topological vector space $\mathbf{R}^{\alpha}$ for $\mathrm{card}(\alpha) \geq \omega$.

Corollary 9.2 The proposition obtained from Corollary 9.1 by interchanging in it the Borel measure by the measure defined on $\overline{B}(E)$(or $\overline{B}(\mathbf{R}^{\alpha})$) is valid also, where by $\overline{B}(E)$ (or $\overline{B}(\mathbf{R}^{\alpha})$) is denoted the σ-algebra of all subsets of E (or $\mathbf{R}^{\alpha}$) with the Baire property.

In sequel we need the following well-known lemmas.

Lemma 9.4 *Let $card(X) = \aleph_1$, and let K be a such class of subsets of X that*
(a) K is σ-ideal;
(b) An union of all elements of K is equal to X;

(c) K contains such an subclass G with card$(G) \leq \aleph_1$ that every element from K is contained in some element from G;

(d) a complement of an arbitrary element from K contains such an element from K whose cardinality is equal to $\aleph_1$.

Then there exists a partition $(X_\alpha)_{\alpha\in I}$ of the set X with card$(I) = \aleph_1$ and

$$(\forall\alpha)(\alpha \in I \to card(X_\alpha) = \aleph_1),$$

such that a subset E belongs to K if and only if E is contained in the union of some countable elements $(X_{\alpha_k})_{k\in N}$ of above-mentioned partition.

Lemma 9.5 *Let card$(X) = \aleph_1$. Let K and L be two classes of subsets of X satisfying the conditions $(a)-(d)$ in Lemma 9.4. Let X be presented as a union of disjoint sets M and N such that $M \in K$ and $N \in L$. Then there exists such a bijection $f : E \to E$ that $f = f^{-1}$ and $f(E) \in L$ if and only if $E \in K$.*

Using lemmas 9.4 and 9.5 we get a following generalization of the famous Erdös-Sierpiński Duality Principle:

Theorem 9.1 *If the Continuum Hypothesis is true, then the strict duality principle between an arbitrary nontrivial continuous σ-finite Borel measure and a Baire category in infinite-dimensional Polish topological vector space is valid.*

Proof. Let μ be an arbitrary continuous σ -finite Borel measure defined on some infinite-dimensional Polish topological vector space X.

We set

$$K = \{M : M \text{ is a first category subset in } X\},$$

$$L = \{N : N \subseteq X \ \&\ N \text{ is of } \mu-\text{measure zero}\}.$$

Note that the class K is generated by the class of a first category F_σ-subsets. The class L is generated by the class of μ-zero G_δ-subsets. Note that cardinalities of generating classes are equal to continuum. Applying Continuum Hypothesis we conclude that classes K and L satisfy the condition (c). The validity of conditions $(a),(b),(d)$ in Lemma 9.4 is obvious. Since X is a Polish space the measure μ is concentrated on the countable union M_0 of compact subsets in X which is a first category subset in X. Obviously, $N_0 = X \setminus L_0$ is of μ-measure zero. Hence, $M_0 \in K,\ N_0 \in L,\ M_0 \cap N_0 = \emptyset,\ M_0 \cup N_0 = X$. The use of Lemma 9.5 completes the proof of Theorem 9.1. □

The following notion is frequently useful in studying various questions of measure theory.

We say that the measure μ defined in a topological vector space (E,T) satisfies the Steinhaus axiom if the following condition

$$(\forall X)(\forall\varepsilon)(X \in \mathrm{dom}(\mu)\ \&\ \mu(X) < \infty\ \&\ \varepsilon > 0 \to$$

$$(\text{there exists a neighborhood } V_\varepsilon \text{ of the zero element } \mathbf{0})\&$$

$$(\forall h)(h \in V_\varepsilon \to \mu((X+h)\triangle X) < \varepsilon))$$

holds.

For $E = R$, we have the following interesting result which is essentially due to Steinhaus (cf.[124],p.42):

Theorem 9.2 *Let X be an arbitrary linear Borel subset in R with a positive Lebesgue measure. Then there exists a positive number ε such that the condition*

$$(\forall x)(x \in \mathbf{R} \;\&\; |x| < \varepsilon \to (x+X) \cap X \textit{ is a second category set})$$

holds.

Note that an arbitrary Haar measure λ defined in the locally compact topological vector space (E,T) satisfies the Steinhaus property(see , e.g. [54]).

Now, we can formulate and prove the following important result for nontrivial σ-finite Borel measures defined in infinite-dimensional Polish topological vector spaces.

Theorem 9.3 *Any nontrivial σ-finite Borel measure defined in the infinite-dimensional Polish topological vector space E does not satisfy the Steinhaus axiom.*

Proof. Let μ be an arbitrary nontrivial σ-finite Borel measure in E. Since every Polish space E is a Radon space (see, e.g. [86],[173]) we can indicate a countable sequence of compact subsets $(K_i)_{i\in N}$ in E such that

$$\mu(E \setminus \cup_{i\in N} K_i) = 0.$$

Using Lemma 9.2, we conclude that for an arbitrary neighborhood V_ε of the zero element $\mathbf{0}$ there exists an element $h_\varepsilon \in V_\varepsilon$ such that the following condition

$$(\cup_{i\in N} K_i + h_\varepsilon) \cap (\cup_{i\in N} K_i) = \emptyset$$

holds. Now, if we consider a Borel set $X \subset (E \cap (\cup_{i\in N} K_i))$ with $0 < \mu(X) < \infty$, then for an arbitrary positive number $\varepsilon < \mu(X)$ and for an arbitrary neighborhood V_ε of the zero element $\mathbf{0}$ we can indicate an element $h_\varepsilon \in V_\varepsilon$ which

$$\mu((X + h_\varepsilon) \triangle X) = \mu(X) > \varepsilon.$$

Hence the measure μ does not satisfy the Steinhaus axiom and Theorem 9.3 is proved. $\square$

Corollary 9.3 Theorem 9.2 remains valid also for an arbitrary nontrivial σ-finite measures defined in the σ-algebra $\overline{B}(E)$ of all subsets of E with a Baire property because such measures are concentrated also on the union of the countable family of compact subsets in E and we can usually repeat the proving scheme implying in Theorem 9.3.

Theorem 9.4 *Any nontrivial σ-finite Borel measure defined on the infinite-dimensional topological vector space $\mathbf{R}^\alpha$ equipped with Tykhonoff topology does not satisfy the axiom of Steinhaus for card$(\alpha) \geq \omega$, where ω denotes the cardinality of all natural numbers.*

Proof. Let μ be an arbitrary nontrivial σ-finite Borel measure defined on $\mathbf{R}^\alpha$. Let us consider any Borel probability measure ν which is equivalent to the measure μ.

Let α_0 be an infinite countable subset of α such that card$(\alpha \setminus \alpha_0) \neq \emptyset$.

Let us define the measure μ_1 by

$$(\forall X)(X \in B(\mathbf{R}^{\alpha_0}) \to \mu_1(X) = \mu(X \times \mathbf{R}^{\alpha \setminus \alpha_0})).$$

Since $\mathbf{R}^{\alpha_0}$ is a Radon space we establish the existence of a countable family of compact subsets $(K_i)_{i \in N}$ in $\mathbf{R}^{\alpha_0}$ such that

$$\mu_1(\mathbf{R}^{\alpha_0} \setminus \bigcup_{i \in N} K_i) = 0.$$

Let us consider a zero vector $\mathbf{0}_\alpha$ in $\mathbf{R}^\alpha$ defined by

$$(\forall i)(i \in \alpha \to \mathbf{0}_\alpha(i) = 0).$$

Let U be an arbitrary neighborhood of the zero vector $\mathbf{0}_\alpha$. Then there exists also a neighborhood V of $\mathbf{0}_\alpha$ having a form

$$V = \prod_{i \in \alpha} V_i,$$

where

$$\mathrm{card}\{i : i \in \alpha \ \& \ V_i \neq \mathbf{R}\} < \omega \ \&$$

$$(\forall i)(i \in \alpha \ \& \ V_i \neq \mathbf{R} \to (\exists a_i)(\exists b_i)(a_i \in \mathbf{R} \ \& \ b_i \in \mathbf{R} \ \& \ a_i < 0 < b_i \ \& \ V_i = (a_i; b_i)).$$

It is clear that the set

$$V_0 = \prod_{i \in \alpha_0} V_i$$

is a neighborhood of the zero vector $\mathbf{0}_{\alpha_0}$. Using Lemma 9.2 we conclude an existence of an element $\overline{h} \in \prod_{i \in \alpha_0} V_i$ such that

$$(\bigcup_{i \in N} K_i) \bigcap (\bigcup_{i \in N} K_i + \overline{h}) = \emptyset.$$

It is clear also that

$$\mu(\mathbf{R}^\alpha \setminus ((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0})) = 0$$

and

$$((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0} + h^*) \bigcap ((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0}) = \emptyset,$$

where

$$h^* = \overline{h} \times g, \ g \in \prod_{i \in \alpha \setminus \alpha_0} V_i.$$

If we consider the set $B \subseteq (\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0}$ with $0 < \mu(B) < \infty$, then for $\varepsilon < \mu(B)$ and for an arbitrary neighborhood U of the zero vector 0_α we can indicate an element $h^* \in U$ such that

$$\mu((B + h^*) \triangle B) = \mu(B) > \varepsilon.$$

This completes the proof of Theorem 9.4. $\square$

Corollary 9.4 One can easily prove that an arbitrary nontrivial σ-finite measure defined on the σ-algebra $\overline{B}(\mathbf{R}^{\alpha})$ of all subsets of $\mathbf{R}^{\alpha}$ with a Baire property does not satisfy the axiom of Steinhaus for $\mathrm{card}(\alpha) \geq \omega$.

Above-mentioned results allow us to conclude that the property of σ-finiteness and the axiom of Steinhaus are not consistent for nontrivial σ-finite Borel measures in infinite-dimensional Polish topological vector spaces. In this connection, the following example is of certain interest.

Example 9.1 Define the measure μ_0 by

$$B \in B(\ell_2) \to \mu_0(B) = \begin{cases} \infty, & \text{if } \mathbf{B} \text{ is of the second category}, \\ 0. & \text{if } \mathbf{B} \text{ is of the first category}, \end{cases}$$

It is proved that, on the one hand, the measure μ_0 satisfies Suslin's property and is invariant with respect to the vector space ℓ_2 (see [17]). On the other hand, using Lemma 9.1, we conclude that the measure μ_0 (unlike the nontrivial σ-finite Borel measures) satisfies the following condition

$$(\forall X)(X \in B(\ell_2) \ \&\ \mu(X) > 0 \to (\exists \varepsilon)(\varepsilon > 0$$

$$\&\ (\forall h)(||h|| < \varepsilon \to (X+h) \cap X \text{ is a second category set}))).$$

This means that the duality between the measure μ_0 (which is not σ-finite) and the Baire category with respect to the sentence (S) is valid in the infinite-dimensional separable Hilbert space ℓ_2. Also note that the measure μ_0 satisfies the axiom of Steinhauss.

Finally, we have the following

Theorem 9.5 *An arbitrary quasi-finite Borel measure μ defined in the infinite-dimensional topological Radon vector space $(V,+)$ does not satisfy the axiom of Steinhauss.*

Proof. Let X be a Borel set with $0 < \mu(X) < \infty$. Since V is a Radon space for a conditional probability measure $\mu(\cdot|X)$ we can indicate such compact subset $F \subseteq X$ that $\mu(F|X) > 0$. Hence, $0 < \mu(F) < \infty$. Let us consider a function $f : \mathbf{R} \times F \times F \to V$ defined by the following formula $f(\alpha, x, y) = \alpha(x-y)$. It is clear that $f(\mathbf{R} \times F \times F) = \cup_{n \in N} f([-n,n] \times F \times F)$. Clearly, $f([-n,n] \times F \times F)$ is the compact subset of V for every $n \in N$. Since V is the second category set we have $V \setminus f(\mathbf{R} \times F \times F) \neq \emptyset$. Let $v \in V \setminus f(\mathbf{R} \times F \times F)$. Let us show that the vector v spans a line L such that every translation of L meets F in at most one point. Indeed, let us assume the contrary and let $y_1, y_2 \in F$ and $y_2 = y_1 + \alpha v$, where $\alpha \neq 0$. Then for the element $v = \frac{1}{\alpha}(y_2 - y_1)$ we can indicate such a natural number n_0, that $v \in f([-n_0, n_0] \times F \times F)$ and we obtain a contradiction. Hence, $F \cap (F + vt) = \emptyset$ for every nonzero parameter $t \in \mathbf{R}$. Now, if we set $\varepsilon = \frac{\mu(F)}{2}$, then for arbitrary neighborhood U_ε of the zero element in V we can indicate a non-zero parameter $t_\varepsilon \in \mathbf{R}$ such that $t_\varepsilon v \in U_\varepsilon$ and $\mu(F \Delta (F + t_\varepsilon v)) > \varepsilon$. This ends the proof of Theorem 9.5. $\square$

Chapter 10

On Strict Transitivity Property

In this chapter we discuss some properties of nontrivial finite Borel measures in infinite-dimensional Polish topological vector spaces. These properties show a significant difference between invariant measure theory in finite-dimensional topological vector spaces and analogous theory in infinite-dimensional topological vector spaces. It is well known that in infinite-dimensional Polish topological vector spaces there are no nontrivial σ -finite Borel measures invariant (moreover, quasi-invariant) with respect to the group of all translations of such spaces. However, there are some nonzero σ -finite Borel measures in these spaces which are invariant (quasi-invariant) with respect to everywhere dense vector subspaces (see, for example, [86],[87]). The main notions and facts which we apply below can be found in [54],[86],[124].

Let (E,τ) be a topological vector space and let U be an arbitrary neighborhood of the zero in E. A translation A_h, defined by $(\forall x)(x \in E \rightarrow A_h(x) = x+h\ )$, is called U-small if $h \in U$.

Let μ be an arbitrary nontrivial σ-finite Borel measure defined on (E,τ) and $P(E)$ be the class of all subsets of E. Let us define a binary relation $\sim$ on $P(E)$ by the formula: $A \sim B$ if and only if $\mu^*(A \triangle B) = 0$. It is easy to see that this relation is an equivalence on $P(E)$. Let $([G]_i)_{i\in I}$ be the family of all equivalence classes. We say that μ is transitively acting measure if, for an arbitrary indices $i \in I$, $j \in I$, there exist $A_i \in [G]_i$, $B_j \in [G]_j$ and $h \in E$, that $A_i + h = B_j$. Similarly, μ is called a strictly transitively acting measure if, for arbitrary indices $i \in I$, $j \in I$, and for an arbitrary neighborhood U of the zero in E, there exist $A_i \in [G]_i$, $B_j \in [G]_j$, and $h \in U$ such that $A_i + h = B_j$.

We say that a topological vector space (E,τ) satisfies the strict transitivity property if an arbitrary σ-finite Borel measure defined in (E,τ) is a strictly transitively acting measure.

In connection with above-mentioned notions, the following examples are of interest.

Example 10.1 Let $\mathbf{R}^n$ be the n-dimensional Euclidean space and b_n be the classical n -dimensional Borel measure on it. If we consider two sets $A = [0;1]^n$ and $B = [0;2]^n$, then we can not indicate sets $A^* \in \mathbf{R}^n$, $B^* \in \mathbf{R}^n$ and an element $h \in \mathbf{R}^n$ such that

$$b_n^*(A^* \triangle A) = 0,\ b_n^*(B^* \triangle B) = 0,\ A^* + h = B^*,$$

where b_n^* denotes the outer measure corresponding to b_n. Indeed, if we assume the contrary that such objects exist, then A^* and B^* must be measurable in the Lebesgue sense and the

condition $A^*+h=B^*$, applying a translation-invariance of the Lebesgue measure $\overline{b_n}$, must imply the validity of the condition $b_n^*(A^*)=b_n^*(B^*)$. The latter relation is not possible, because $b_n^*(A^*)=b_n^*([0;1]^n)=1<2^n=b_n^*([0;2]^n)=b_n^*(B^*)$. Hence, b_n is not a transitively acting measure.

Example 10.2 Let μ be a canonical n-dimensional Gaussian measure defined on $\mathbf{R}^n$. It is well known that the completion $\overline{\mu}$ of the measure μ is quasi-invariant under the group of all translations of $\mathbf{R}^n$, i.e.,

$$(\forall h)(\forall X)(h\in\mathbf{R}^n \ \& \ X\in \mathrm{dom}(\overline{\mu})\rightarrow(\overline{\mu}(X)>0 \ \text{ if and only if } \ \overline{\mu}(X+h)>0)).$$

Let A and B be such Borel subsets in $\mathbf{R}^n$ that $\mu(A)>0,\ \mu(B)=0$. We state that μ is not a transitively acting measure. Indeed, if we assume the contrary, then there exist two subsets A^* and B^* of $\mathbf{R}^n$ and an element $h\in\mathbf{R}^n$ such that

$$\overline{\mu}(A^*\triangle A)=0,\ \overline{\mu}(B^*\triangle B)=0,\ A^*+h=B^*.$$

Hence, $\overline{\mu}(A^*)=\mu(A)>0$. From the quasi-invariance of the measure $\overline{\mu}$, we obtain $\overline{\mu}(A^*+h)>0$.

On the other hand, we have

$$\overline{\mu}(A^*+h)=\overline{\mu}(B^*)=\overline{\mu}(B)=0,$$

which contradicts the condition $\overline{\mu}(A^*+h)>0$.

The following example shows us that the class of all σ-finite nonatomic strictly transitively acting Borel measures on $\mathbf{R}^n$ is not empty.

Example 10.3 Let $(q_k)_{k\in N}$ be a sequence of all rational numbers. Let us define a σ-finite Borel measure μ by the following formula :

$$(\forall X)(X\in B(\mathbf{R}^n)\rightarrow\mu(X)=\sum_{k\in N}b_{n-1}(X\cap(\mathbf{R}^{n-1}\times\{q_k\}))).$$

One can easily demonstrate that μ is strictly transitively acting.

The next well-known assertions play a major role in our further consideration.

Lemma 10.1(Ulam) *Let E be an infinite-dimensional Polish topological vector space. Then, for an arbitrary nontrivial σ-finite Borel measure, there exists a countable family of compact subsets $(K_i)_{i\in N}$ in E such that*

$$\mu(E\setminus\bigcup_{i\in N}K_i)=0.$$

Lemma 10.2(Baire) *An arbitrary nonempty Polish space is a space of the second category.*

The proof of lemmas 10.1,10.2 can be found in [86],[173].

Remark 10.1 We recall the well-known consequence of lemmas 10.1,10.2 stating that, for an arbitrary infinite-dimensional Polish topological vector space E, the following representation $E=X\cup Y$ is valid, where X is a set of first category and Y is a set of μ^*-measure zero.

Remark 10.2 Let E be an infinite-dimensional Polish topological vector space. Using Lemma 10.3, we establish that for an arbitrary nontrivial σ- finite Borel measure, there exists a subset Y of μ^*-measure zero with the following property: for an arbitrary neighborhood U of $0 \in E$, there exists $h \in U$ such that $E = Y \cup (Y+h)$. (Analogous representation of E by the set of first category is not possible, because any translation preserves the category and E is the set of second category.) Hence, for an arbitrary set $A \subset E$ and for an arbitrary neighborhood U of $0 \in E$, there exist two subsets $X \subset E, Y \subset E$ of μ^* -measure zero and $h \in E$ such that $A = X \cup (Y+h)$. The last relation means that the minimal translation-invariant σ-algebra generated by the σ-ideal of all subsets of μ^*-measure zero, coincides with $P(E)$. Note that analogous result is not valid for an arbitrary nonzero σ-finite Borel measure defined in the finite-dimensional Polish topological vector space.

Remark 10.3 Remark that no translation-invariant measure (in Solovay's model) defined on the class of all subsets of $\mathbf{R}^n$ is a strictly transitively acting measure.

Applying lemmas 9.2,10.1 and 10.2, we are able to prove the following

Theorem 10.1 *Any infinite-dimensional Polish topological vector space H has the strict transitivity property.*

Proof. Let μ be an arbitrary nontrivial σ-finite Borel measure in H. Let A and B be two subsets in H. Using Lemma 10.1, we conclude the existence of such a countable family of compact subsets $(K_i)_{i \in N}$ in H that

$$\mu(H \setminus \bigcup_{i \in N} K_i) = 0.$$

Let U be an arbitrary neighborhood of the zero element. Using Lemma 9.2 we conclude the existence of an element $h_0 \in U$ such that

$$(\bigcup_{i \in N} K_i + h_0) \bigcap (\bigcup_{i \in N} K_i) = \emptyset.$$

Let

$$A^* = \overline{A} \bigcup (\overline{B} - h_0), B^* = \overline{B} \bigcup (\overline{A} + h_0),\ h^* = -h_0,$$

where $\overline{A} = A \bigcap \bigcup_{i \in N} K_i$ and $\overline{B} = B \bigcap \bigcup_{i \in N} K_i$.

We have

$$\mu^*(A \triangle A^*) = \mu^*(\{(A \triangle A^*) \cap (\bigcup_{n \in N} K_n)\} \cup \{(A \triangle A^*) \cap (H \setminus \bigcup_{n \in N} K_n)\}) \leq$$

$$\leq \mu^*((A \triangle A^*) \cap (\bigcup_{n \in N} K_n)) + \mu^*((A \triangle A^*) \cap (H \setminus \bigcup_{n \in N} K_n)) \leq$$

$$\leq \mu^*((A \triangle A^*) \cap (\bigcup_{n \in N} K_n)) \leq \mu^*((A \setminus A^*) \cap (\bigcup_{n \in N} K_n)) + \mu^*((A^* \setminus A) \cap (\bigcup_{n \in N} K_n)) =$$

$$= \mu^*((A \cap (\bigcup_{n \in N} K_n)) \setminus (A^* \cap (\bigcup_{n \in N} K_n))) + \mu^*((A^* \cap (\bigcup_{n \in N} K_n)) \setminus (A \cap (\bigcup_{n \in N} K_n))) =$$

$$= \mu^*(\overline{A} \setminus \overline{A}) + \mu^*(\overline{A} \setminus \overline{A}) = \mu^*(\emptyset) + \mu^*(\emptyset) = 0.$$

Using the same argument, we can prove that

$$\mu^*(B^* \triangle B) = 0.$$

Finally, we have

$$A^* + h_0 = (\overline{A} \bigcup (\overline{B} - h_0)) + h_0 = (\overline{A} + h_0) \bigcup ((\overline{B} - h_0) + h_0) = \overline{B} \bigcup (\overline{A} + h_0) = B^*.$$

Theorem 10.1 is proved. □

Corollary 10.1 There is no a probability Borel measure in H, such that $S \subset H$ being shy is equivalent to $\mu(S) = 0$ (cf.[29], p.119).

The next assertion describes situations when the vector space R^{α} also has an analogous property.

Theorem 10.2 *A vector spaces $\mathbf{R}^{\alpha}$ equipped with Tykhonoff topology has a strict transitivity property if and only if $card(\alpha) \geq \omega$, where ω denotes the cardinality of all natural numbers set.*

Proof. Necessity. Using Example 10.1, the proof of necessity is trivial.

Sufficient. Let $\mathrm{card}(\alpha) \geq \omega$. Let us show that a topological vector spaces $\mathbf{R}^{\alpha}$ has the strict transitivity property. Let μ is an arbitrary nontrivial σ-finite Borel measure defined in $\mathbf{R}^{\alpha}$. Without loss of generality we can assume that the measure μ is a probability measure. Really, if we prove that an equivalent Borel probability measure ν is a strictly transitively acting measure then from the equivalence of measures ν and μ we automatically get that the measure μ is also a strictly transitively acting measure.

Let α_0 be a such infinite countable subset of α that $\mathrm{card}(\alpha \setminus \alpha_0) \neq \emptyset$.

Let us define the measure μ_1 by

$$(\forall X)(X \in B(\mathbf{R}^{\alpha_0}) \rightarrow \mu_1(X) = \mu(X \times \mathbf{R}^{\alpha \setminus \alpha_0})).$$

As $\mathbf{R}^{\alpha_0}$ is a Polish space, by Lemma 10.1 we conclude the existence of a countable family of compact subsets $(K_i)_{i \in N}$ such that

$$\mu_1(\mathbf{R}^{\alpha_0} \setminus \bigcup_{i \in N} K_i) = 0.$$

Let us consider a zero element $\mathbf{0}_{\alpha} \in \mathbf{R}^{\alpha}$, defined by

$$(\forall i)(i \in \alpha \rightarrow \mathbf{0}_{\alpha}(i) = 0).$$

Let V be an arbitrary neighborhood of the zero element $\mathbf{0}_{\alpha}$. Then there exists also a neighborhood $U \subset V$ of $\mathbf{0}_{\alpha}$ having a form

$$U = \prod_{i \in \alpha} U_i,$$

where

$$\mathrm{card}\{i : i \in \alpha \ \&\ U_i \neq \mathbf{R}\} < \omega \ \&$$

$$(\forall i)(i \in \alpha \ \&\ U_i \neq \mathbf{R} \rightarrow (\exists a_i)(\exists b_i)(a_i \in R \ \&\ b_i \in R \ \&\ a_i < 0 < b_i \rightarrow U_i = (a_i; b_i)).$$

It is clear that the set

$$U_0 = \prod_{i \in \alpha_0} U_i$$

is a neighborhood of the zero element $\mathbf{0}_{\alpha_0}$ in $\mathbf{R}^{\alpha_0}$. Using Lemma 9.3, we conclude an existence of an element $\overline{h} \in \prod_{i \in \alpha_0} U_i$ such that

$$(\bigcup_{i \in N} K_i) \bigcap (\bigcup_{i \in N} K_i + \overline{h}) = \emptyset.$$

Note that

$$\mu(\mathbf{R}^{\alpha} \setminus ((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0})) = 0$$

and

$$((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0} + h^*) \bigcap ((\bigcup_{i \in N} K_i) \times \mathbf{R}^{\alpha \setminus \alpha_0}) = \emptyset,$$

where

$$h^* = \overline{h} \times g, \; g \in \prod_{i \in \alpha \setminus \alpha_0} U_i.$$

By this method used in the process of proving Theorem 10.1, we easily prove that the measure μ is a strictly transitively acting measure.

As a σ-finite Borel measure was taken arbitrarily, the sufficiency of Theorem 10.2 is proved. $\square$

Remark 10.4 Note that the vector space $\mathbf{R}^{\alpha}$ is not a Radon space for $\text{card}(\alpha) > \omega$.

In connection with theorems 10.1,10.2 and Remark 10.4, the following problem is interesting:

Give a description of the class of all topological vector spaces with the strict transitivity property.

Finally, we discuss again a duality between the measure and the Baire category in infinite-dimensional topological vector spaces.

Theorem 10.3 *The duality between an arbitrary nontrivial σ-finite Borel measure and the Baire category with respect to the sentence P_0, defined by*

$$(\forall A)(\forall B)(\forall U)(A \subset E \;\&\; B \subset E \;\&\; U \textit{ is an arbitrary neighborhood}$$

$$\textit{of the zero of } E \;\Rightarrow (\exists A^*)(\exists B^*)(\exists h^*)(A^* \subset E \;\&\; A \triangle A^* \textit{is small in the}$$

$$\textit{sense of measure} \;\&\; B^* \subset E \;\&\; B \triangle B^* \textit{is small in the sense of measure} \;\&$$

$$h^* \in U \rightarrow A^* + h^* = B^*),$$

is not valid in the infinite-dimensional Polish topological vector space E.

Proof. Note that categories of subsets in an infinite-dimensional Polish topological vector space E are invariant under translations. On the other hand, if the symmetric sum

$X \triangle Y$ is small in the sense of category then each of them has the same category. If A is a first category subset of E and B is a second category subset of E then we can not indicate two subsets $A^* \subseteq E$ and $B^* \subseteq E$ and an element $h \in E$ such that both $(A \triangle A^*)$ and $(B \triangle B^*)$ are small in the sense of category and the equality $A^* + h = B^*$ is valid. Hence, the sentence P_0 is not valid. On the other hand, Theorem 10.1 is just the validity of the sentence P_0 for an arbitrary non-trivial σ-finite Borel measure defined in infinite-dimensional Polish topological vector space E.

This ends the proof of Theorem 10.3. $\square$

Chapter 11

Invariant Extensions of Haar Measures

Some methods of combinatorial set theory have lately been successfully used in different areas of mathematics, for example, in topology and measure theory. Among them, special mention should be made of the method of constructing a maximal (in the sense of cardinality) family of independent families of sets in arbitrary infinite base spaces. The question of the existence of a maximal (in the sense of cardinality) independent family of subsets was considered by A. Tarski (see e.g.[107]). He proved that this cardinality is equal to $2^{card(E)}$. This result found an interesting application in general topology by means of which it was proved that in an arbitrary infinite space E the cardinality of the class of all ultrafilters is equal to $2^{2^{card(E)}}$ (see, e.g.[84]). Using the method of an independent family in the case of the Euclidean space E_n, A.B. Kharazishvili constructed a maximal (in the sense of cardinality) family of orthogonal elementary D_n-invariant extensions of the Lebesgue measure (see [85]).

E. Szpilrajn (E. Marczewski) was the first who suggested the method of constructing nonseparable extensions of the Lebesgue measure (see [169]). Later, S. Kakutani, K. Kodaira and J. Oxtoby constructed nonseparable invariant extensions of the Lebesgue measure (see [80], [102]). We must say that above-mentioned methods can be successfully generalized to some classes of topological groups (see e.g. [85]).

In [85], the method of an independent family of sets is used to construct an example of a nonelementary D_n-invariant extension μ of the Lebesgue measure l_n such that the topological weight $a(\mu)$ of the metric space $(\mathrm{dom}(\mu),\rho_\mu)$ associated with the measure μ is maximal; in particular, this cardinality is equal to the cardinal number 2^c, where c denotes the cardinality of the continuum.

In this chapter, the method of an independent family of sets due to A.Tarski, is generalized and successfully used to construct a maximal family of orthogonal invariant (elementary and nonelementary) extensions of the Haar measure defined on an arbitrary uncountable locally-compact σ-compact topological group. One method for construction of nonelementary invariant extensions of the Haar measure is considered, by means of which one problem formulated in [85] is solved.

Let us recall some definitions which are used in this chapter.

Let E be the main base space and β be some cardinal number.

We say that a family $(X_i)_{i\in I}$ of subsets of the set E is β-independent if the condition

$$(\forall J)(J \subset I \ \&\ \mathrm{card}(J) < \beta \to \bigcap_{i\in J} \overline{X}_i \neq \emptyset),$$

holds, where

$$(\forall i)(i \in I \to (\overline{X}_i = X_i) \vee (\overline{X}_i = (E \setminus X_i))).$$

We say that a family $(X_i)_{i\in I}$ of subsets of the space E is strictly β-independent if the condition

$$(\forall J)(J \subset I \ \&\ card(J) \leq \beta \to \bigcap_{i\in J} \overline{X}_i \neq \emptyset)$$

holds, where

$$(\forall i)(i \in I \to (\overline{X}_i = X_i) \vee (\overline{X}_i = E \setminus X_i)).$$

Remark 11.1 It is clear that the β-independence of the family $(X_i)_{i\in I}$ does not imply its strictly β-independence.

The question of the existence of an $\aleph_0$-independent family of subsets in an arbitrary infinite space E, with maximal cardinality, was solved by A.Tarski. He proved that this power is equal to $2^{2^{card(E)}}$. This result gave rise to many interesting applications, in particular, it was proved in general topology that the cardinality of all ultrafilters defined in an arbitrary infinite space E is equal to $2^{2^{card(E)}}$.

If the cardinality of the base space E is equal to the continuum, one can prove the existence of a strictly $\aleph_0$-independent family of subsets of E with a maximal possible cardinality 2^c, where c is the cardinality of the continuum. This result found applications in the theory of extensions of the Lebesgue measure, in particular, it was used to construct D_n-invariant extensions of the Lebesgue measure.

Let us consider the problem of the existence of a maximal (in the sense of cardinality) strictly independent family of subsets in an arbitrary infinite space.

The next auxiliary proposition plays the key role in this chapter.

Theorem 11.1 *If an infinite set E satisfies the condition*

$$Card(E^{\beta}) = card(E),$$

where β is an infinite cardinal number, then there exists a maximal (in the sense of cardinality) strictly β-independent family $(X_i)_{i\in I}$ of subsets of the space E, such that

$$card(I) = 2^{card(E)}.$$

Proof. Let us prove the theorem in four steps.

I. Assume that X is a set such that $card(X) = card(E)$. Let $(X_i)_{i\in I}$ be a partition of the set X such that

$$card(I) = card(E),$$

$$(\forall i)(i \in I \to card(X_i) = card(E)).$$

Let us consider the class Ω' of subsets of X defined by

$$\Omega' = \Big\{Y \mid Y \subseteq X \;\&\; (\forall i)(i \in I \to card(Y \cap X_i) = 1)\Big\}.$$

The class Ω' satisfies the conditions

$$card(\Omega') = card(\mathcal{B}(E)),$$

$$(\forall Y)(\forall Z)(Y \in \Omega' \;\&\; Z \in \Omega' \;\&\; Y \neq Z \to Y \setminus Z \neq \emptyset).$$

II. Assume that $(X_i)_{i \in I}$ is a partition of X such that

$$card(I) = card(E),$$
$$(\forall i)(i \in I \to card(X_i) = card(E)).$$

For an arbitrary index $i \in I$, denote by Ω'_i a class of all subsets of X_i such that

$$card(\Omega'_i) = card(\mathcal{B}(E)),$$

$$(\forall Y)(\forall Z)(Y \in \Omega'_i \;\&\; Z \in \Omega'_i \;\&\; Y \neq Z \to Y \setminus Z \neq \emptyset).$$

Fix the set J with $card(J) = card(\mathcal{B}(E))$ and consider a representation $(Y_{ij})_{j \in J}$ of the class Ω'_i such that

$$(\forall j)(j \in J \to Y_{ij} \in \Omega'_i).$$

Let us put

$$\Omega'' = \Big\{Y \mid (\exists j)(j \in J \;\&\; Y = \bigcup_{i \in I} Y_{ij}\Big\}.$$

It is easy to verify that the class Ω'' satisfies the condition

$$card(\Omega'') = card(\mathcal{B}(E)),$$

$$(\forall Y)(\forall Z)(Y \in \Omega'' \;\&\; Z \in \Omega'' \;\&\; Y \neq Z \to$$
$$\to card(Y \setminus Z) = card(E)).$$

III. Denote by U the class of subsets of Y defined by
$U = \{Z \mid (Y \in \Omega'' \;\&\; Z$ is a class of all β-powerful subsets of the set $Y\}$.
Assume that $(Z_\xi)_{0 \le \xi \le \omega_\beta}$ is an arbitrary sequence of elements of U. Let us prove that

$$card(Z_0 \setminus \bigcup_{\xi \in]0;\omega_\beta[} Z_\xi) \ge card(E).$$

Let $(Y_\xi)_{0 \le \xi \le \omega_\beta}$ be a sequence from the class Ω'' which corresponds to the sequence $(Z_\xi)_{0 \le \xi \le \omega_\beta}$.

Since the power of the class Ω'' is equal to $2^{card(E)}$, the sequence $(Y_\xi)_{0 \le \xi \le \omega_\beta}$ can be continued as for the sequence $(Y_\xi)_{\xi < \omega_E}$ of elements of the class Ω''.

Using the method of transfinite induction, construct a family of different elements $(x_\xi)_{\xi < \omega_E}$ such that

$$(\forall \xi)(\xi < \omega_E \to x_\xi \in Y_0 \setminus Y_\xi).$$

Assume that, for an ordinal $\xi < \eta$, the sequence $(x_\xi)_{\xi<\eta}$ is already constructed. We can define the element x_η by

$$x_\eta = \tau_x(Y_0 \setminus (Y_\eta \cup (\bigcup_{\xi<\eta} x_\xi)))$$

and therefore the sequence $(x_\xi)_{\xi<\omega_E}$ is constructed.

Let us consider

$$(\bigcup_{0\le\xi\le\omega_\beta} x_\xi) \cup (\bigcup_{\zeta\in K} x_\zeta),$$

where

$$K \subseteq]\omega_\beta;\omega_E[\text{ and } card(K) = \beta.$$

It is easy to verify that this element belongs to the difference

$$Z_0 \setminus \bigcup_{0<\xi<\omega_\beta} Z_\xi.$$

If we consider every β-powerful subset of the set $]\omega_\beta;\omega_E[$, then we obtain

$$card(Z_0 \setminus \bigcup_{0<\xi<\omega_\beta} Z_\xi) \ge card(E).$$

As the power of the class of all β-powerful subsets of the set E is $Card(E)$, we conclude that there exists a class Ω''' of subsets of E, such that:

a) $card(\Omega''') = card(\mathcal{B}(E))$;

b) for an arbitrary ω_β-sequence $(Z_\xi)_{\xi\in]0;\omega_\beta[}$ of elements of the class Ω''' the condition

$$card(Z_0 \setminus \bigcup_{\xi\in]0;\omega_\beta[} Z_\xi) \ge card(E)$$

holds.

IV. Let us put

$$\mathcal{E}(E) = \Big\{Y | Y \in \mathcal{B}(E) \ \& \ card(Y) \le \beta\Big\},$$

$\Omega'''' = \{Z|(\exists Y)(Y \in \Omega''' \ \& \ Z$ is a class of all β-powerful subsets of the set E, such that its elements intersect the set $Y)\}$.

We shall now prove that, for every definite family of elements $(Z_\xi)_{\xi\in[0;\omega_\beta]}$ of the class $\Omega''' \subseteq \mathcal{B}(\mathcal{E}(E))$, the intersection

$$\bigcap_{\xi\le\omega_\beta} \overline{Z}_\xi,$$

where $\overline{Z}_\xi = Z_\xi \vee \overline{Z}_\xi = \mathcal{E}(E) \setminus Z_\xi$, has the power $Card(E)$.

Consider

$$\bigcap_{\xi\le\omega_\beta} \overline{Z}_\xi = (\bigcap_{\xi\in P} Z_\xi) \cap (\bigcap_{\eta\in[0;\omega_\beta[\setminus P} (\mathcal{E}(E) \setminus Z_\eta)).$$

Let us construct two sequences

$$(Z^{(1)}_{\xi})_{\xi\in P} \textit{ and } (Z^{(2)}_{\eta})_{\eta\in[o;\omega_E[}$$

such that

$$1)(\forall\xi)(\forall\eta)(\xi\in P \;\&\; \eta\in[0;\omega_E[\rightarrow Z^{(1)}_{\xi}\neq Z^{(2)}_{\eta});$$

$$2)(\forall\xi)(\xi\in P\rightarrow Z^{(1)}_{\xi}\in Y_{\xi}\setminus\bigcup_{\eta\in[0;\omega_\beta]\setminus P} Y_{\eta});$$

$$3)(\forall\eta)(\eta\in[0;\omega_E[\rightarrow Z^{(2)}_{\eta}\in\bigcup_{\eta\in[0;\omega_\beta]\setminus P} Y_{\eta}).$$

Note that

$$Z_{\xi}=\Big\{Y|Y\subseteq E \;\&\; Card(Y)=\beta \;\&\; Y\cap Y_{\xi}\neq\emptyset\Big\},$$

$$\mathcal{E}(E)\setminus Z_{\xi}=\Big\{Y|Y\subseteq E \;\&\; Card(Y)=\beta \;\&\; Y\cap Y_{\xi}\neq\emptyset\Big\}.$$

Let us construct the sequence $(Z^{(1)}_{\xi})_{\xi\in P}$.

Assume that, for an ordinal η, the sequence $(Z^{(1)}_{J})_{J<\eta}$ is constructed.

If we put

$$Z^{(1)}_{\eta}=\tau_x(Y_{\eta}\setminus((\bigcup_{\xi\in[0;\omega_\beta[\setminus P} Y_{\xi})\cup(\bigcup_{J<\eta} Z^{(1)}_{J}))),$$

then the sequence $(Z^{(1)}_{\xi})_{\xi\in P}$ is constructed.

As the cardinality of the set $E\setminus\bigcup_{\eta\in[0;\omega_\beta]} Y_{\eta}$ is equal to $Card(E)$, we can choose a sequence $(Z^{(2)}_{\eta})_{\eta\in[0;\omega_E]}$ such that

$$(\forall\eta)(\eta\in[0;\omega_E[\rightarrow Z^{(2)}_{\eta}\in E\setminus\bigcup_{\eta\in[0;\omega_\beta]} Y_{\eta}).$$

Now, it is easy to prove that the sequences $(Z^{(1)}_{\xi})_{\xi\in P}$ and $(Z^{(2)}_{\eta})_{\eta\in[o;\omega_E[}$ satisfy all the conditions mentioned above.

It is clear that the set

$$(\bigcup_{\xi\in P} Z^{(1)}_{\xi})\cup(\bigcup_{J\in K} Z^{(2)}_{J})$$

intersects every element $Y_{\xi}(\xi\in P)$ and belongs to the set $\bigcap_{\xi\in P} Z_{\xi}$, where

$$K\subseteq[0;\omega_E[\textit{ and } Card(K)=\beta.$$

Let us remark that, for $\eta\in[0;\omega_\beta[\setminus P$, the set

$$(\bigcup_{\xi\in P} Z^{(1)}_{\xi})\cup(\bigcup_{J\in K} Z^{(2)}_{J})$$

does not intersect the set Y_{η}.

This means that

$$(\bigcup_{\xi\in P} Z_{\xi}^{(1)}) \cup (\bigcup_{J\in K} Z_{J}^{(2)}) \in \bigcap_{\xi\in[0;\omega_\beta[\setminus P} (\mathcal{E}(E)\setminus Z_{\xi}).$$

If we consider different meanings of the set K, we obtain a class of all β-powerful subsets of the set E whose cardinality is equal to $card(E)$, because $card(E^{\beta}) = card(E)$. Hence, we have

$$card((\bigcap_{\xi\in P} Z_{\xi}) \cap (\bigcap_{\eta\in[0;\omega_\beta[\setminus P} (\mathcal{E}(E)\setminus Z_{\eta}))) = card(E).$$

Let us denote by f the one-to-one mapping between the class $\mathcal{E}(E)$ and the set E.

Now if we consider the class $\Omega = f(\Omega'''')$, then Theorem 11.1 will be proved. □

Let κ and λ be two infinite cardinal numbers. Following [25, p. 20], the cardinal number λ is is said to be cofinal with κ if there exists a transfinite sequence $(\alpha_\zeta)_{\zeta\in\lambda}$ of ordinals such that

$$\alpha_\zeta < \kappa(\zeta\in\lambda),$$

$$\sup_{\zeta}(\alpha_\zeta) = \phi_\kappa,$$

where ϕ_κ denotes a first ordinal number of the cardinality κ.

We say that a cardinal number λ is the cofinality of κ(and write $\lambda = cf(\kappa)$) if λ is the least cardinal cofinal with κ. Of course, the equality $cf(\kappa) \leq \kappa$ holds.

Lemma 11.1 *Assume that the Generalized Continuum Hypothesis holds, and let* α *be an infinite cardinal. If* β *is a cardinal number such that*

$$\beta < \alpha \text{ and } \alpha^{\beta} \neq \alpha,$$

then the cardinal number β *is cofinal with the cardinal number* α*.*

Proof. Let us assume to the contrary that the cardinal number β is not cofinal with the cardinal number α. Let ϕ_α be a first ordinal number of a cardinality α. Then, for an arbitrary transfinite sequence of ordinals $(\eta_i)_{i\in\beta}$ satisfying the condition $(\forall i)(i\in\beta \to \eta_i < \phi_\alpha)$ there exists an ordinal number ξ such that

(a) $\xi < \phi_\alpha$,

(b) $(\forall i)(i\in\beta \to \eta_i \leq \xi)$.

For an arbitrary ordinal number $\xi \in [0;\phi_\alpha[$, denote by K_ξ the class of all transfinite β-sequences of ordinal numbers from the segment $[0;\xi]$. It is clear that the cardinality of the class K_ξ is equal to $(\mathrm{card}(\xi))^{\beta}$.

Let K be a class of all transfinite β-sequences of ordinal numbers from the segment $[0;\phi_\alpha[$. It is clear that $\mathrm{card}(K) = \alpha^{\beta}$.

Since β is not cofinal with α, we have the following representation

$$K = \bigcup_{\xi\in[0;\alpha[} K_\xi,$$

which implies

$$card(K) = card(\bigcup_{\xi \in [0,\alpha[} K_\xi).$$

Under Generalized Continuum Hypothesis we have $2^{\max\{\mathrm{card}(\xi),\beta\}} \leq \alpha$ for $\xi < \phi_\alpha$. Hence, for each class K_ξ, we have

$$\mathrm{card}(K_\xi) = (\mathrm{card}(\xi))^\beta \leq (2^{\mathrm{card}(\xi)})^\beta = 2^{\max\{\mathrm{card}(\xi),\beta\}} \leq \alpha.$$

Finally, we have

$$\alpha^\beta = \mathrm{card}(K) = \mathrm{card}(\cup_{0 \leq \xi < \phi_\alpha} K_\xi) \leq \sum_{0 \leq \xi < \phi_\alpha} (\mathrm{card}(\xi))^\beta \leq \mathrm{card}(\alpha \times \alpha) = \alpha.$$

The latter equality is a required contradiction and Lemma 11.1 is proved. □

Let us consider some applications of the results obtained above.

The following proposition is valid.

Lemma 11.2 *Let H be an arbitrary locally compact σ-compact topological group, λ be the Haar measure defined on the group H and let α be an arbitrary cardinal number such that:*

$$\alpha \leq card(H).$$

Then there exists a family $(X_i)_{i \in I}$ of subsets of the set H such that:
1) $Card(I) = \alpha$;
2) $(\forall i)(\forall i')(i \in I \ \&\ i' \in I \ \&\ i \neq i' \to X_i \cap X_{i'} = \emptyset)$;
3) $(\forall i)(\forall F)(i \in I \ \&\ (F$ *is a closed subset of the space H with* $\lambda(F) > 0) \to$ $card(X_i \cap F) = card(H))$;
4) $(\forall I')(\forall g)(I' \subseteq I \ \&\ g \in H \to card(g(\bigcup_{i \in I'} X_i) \triangle (\bigcup_{i \in I'} X_i)) < card(H))$.

Proof. Without loss of generality,we can assume that $\alpha = Card(H)$.

Let us denote by ω_H the first $Card(H)$-powerful ordinal number. Let us consider a partition $(\Xi_i)_{i \in I}$ of the segment $[0; \varphi_H[$ such that

(A) $card(I) = card(H)$,

(B) $(\forall i)(i \in I \to card(\Xi_i) = card(H))$.

It is easy to prove an existence of a family $(F_\xi)_{\xi < \omega_H}$ of subsets of the group H such that $(F_\xi)_{\xi \in \Xi_i}$ is a family of all closed subsets of the space H for all $i \in I$, where every such a subset F_ξ appears $card(H)$-times in the sequence $(F_\xi)_{\xi \in \Xi_i}$ and has a positive λ-measure.

Assume that $(g_\xi)_{\xi < \omega_H}$ is a ω_H-sequence of all elements of the group H. For an arbitrary ordinal number $\xi < \omega_H$, denote by H_ξ the subgroup of H generated by the ξ-sequence $(g_\eta)_{\eta < \xi}$. Using the method of transfinite recursion, define a family $(Y_\xi)_{\xi < \omega_H}$ of subsets of the space H such that the following conditions hold:

(a) $(\forall \xi)(\forall \eta)(\xi < \omega_H \ \&\ \xi \neq \eta \to Y_\xi \cap Y_\eta = \emptyset)$.

(b) $(\forall \xi)(\xi < \omega_H \to Y_\xi \cap F_\xi \neq \emptyset)$.

(c) $(\forall \xi)(\xi < \omega_H \to Y_\xi$ is the class of intransitivity of the group $H_\xi)$.

Assume that, for an ordinal number $\xi < \omega_H$, a subsequence $(Y_\eta)_{\eta < \xi}$ is constructed such that the conditions $(a), (b), (c)$ are fulfilled.

Since the relation

$$card(H_\xi(\bigcup_{\eta<\xi} Y_\eta)) \leq card(\xi) + \aleph_0$$

is true, we have

$$card(F_\xi \setminus H_\xi(\bigcup_{\eta<\xi} Y_\eta)) = card(H).$$

Let us put

$$y_\xi = \tau_x(F_\xi \setminus H_\xi(\bigcup_{J<\xi} Y_\eta)),\ Y_\xi = H_\xi(y_\xi),$$

where τ_x is the global operator of choice.

Hence the sequence $(Y_\xi)_{\xi<\omega_H}$ is defined.

Let us denote by X_i the union of elements of the family $(Y_\xi)_{\xi\in\Xi_i}$ for $i \in I$.

It is clear that the family $(X_i)_{i\in I}$ satisfies conditions $1)-4)$, and Lemma 11.2 is proved. $\square$

It is not difficult to prove the next proposition.

Lemma 11.3 *Let K be an H-invariant σ-ideal in the local compact σ-compact topological group H such that*

$$(\forall Z)(Z \in K \to \lambda_*(Z) = 0).$$

Then the functional μ defined by

$$\mu((X \cup Z') \cup Z'') = \lambda(X),$$

where X is a Borel subset of H and Z' and Z'' are elements of the ideal K, is an H-invariant extension of the Haar measure λ.

For the proof, see, e.g., [85], [92].

Theorem 11.2 *Let H be a locally compact σ-compact topological group such that $card(H^{\aleph_0}) = card(H)$. Then there exists a family $(\mu_t)_{t\in T}$ of measures defined on some σ-algebra of subsets of H such that:*

1) $card(T) = 2^{2^{card(H)}}$;

2) $(\forall t)(t \in T \to$ *the measure* μ_t *is an H-invariant extension of the Haar measure* $\lambda)$;

3) $(\forall t)(\forall t')(t \in T\ \&\ t' \in T\ \&\ t \neq t' \to \mu_t$ *and* $\mu_{t'}$ *are orthogonal* [1] *measures*$)$.

We will need the following lemma.

Lemma 11.4 *Let H be a locally compact σ-compact topological group such that*

$$card(H^{\aleph_0}) = card(H).$$

Then there exists a family $(A_j)_{j\in J}$ of subsets of the group H such that:

1) $card(J) = 2^{card(H)}$;

[1] μ_t and $\mu_{t'}$ are called orthogonal if there exists $X \in S(H)$ such that $\mu_t(X) = 0$ and $\mu_{t'}(H \setminus X) = 0$.

2) *for an arbitrary sequence* $(j_k)_{k\in N}$ *from indices from the set* J*, the intersection*

$$\bigcap_{k\in N}\overline{A}_{j_k},$$

where

$$(\forall k)(k\in N\rightarrow \overline{A}_{j_k}=A_{j_k}\vee \overline{A}_{j_k}=H\setminus A_{j_k}),$$

is an H*-invariant subset of the group* H*;*

3) *for an arbitrary sequence* $(j_k)_{k\in N}$ *of indices from the set* J *and for an arbitrary closed subset* $F\subseteq H$ *with* $\lambda(F)>0$*, the condition*

$$card((\bigcap_{k\in N}\overline{A}_{j_k})\cap F)=card(H)$$

holds, where

$$(\forall k)(k\in N\rightarrow \overline{A}_{j_k}=A_{j_k}\vee \overline{A}_{j_k}=H\setminus A_{j_k}).$$

Proof. Let us consider a family $(X_i)_{i\in I}$ of subsets of the group H such that the following conditions hold:

1) $card(I)=card(H)$;
2) $(\forall i)(\forall i')(i\in I\ \&\ i'\in I\ \&\ i\neq i'\rightarrow X_i\cap X_i=\emptyset)$;
3) $(\forall i)(\forall F)(i\in I\ \&\ (F$ is a closed subset of the group H with $\lambda(H)>0)\rightarrow card(X_i\cap F)=card(H))$;
4) $(\forall I')(\forall g)(I'\subseteq I\ \&\ g\in H\rightarrow card(g(\bigcup_{i\in I'}X_i)\triangle(\bigcup_{i\in I'}X_i))<card(H))$.

The existence of such a family is guaranteed by Lemma 11.1.

Let $(I_j)_{j\in J}$ be a family of subsets of I, such that:

a) $card(J)=2^{card(H)}$;
b) for an arbitrary sequence $(j_k)_{k\in N}$ of indices from the set J we have

$$\bigcap_{k\in N}\bar{I}_k\neq\emptyset,$$

where

$$(\forall k)(k\in N\rightarrow \bar{I}_{j_k}=I_{j_k}\vee \bar{I}_{j_k}=I\setminus I_{j_k}).$$

Note that the existence of such a family is guaranteed by Theorem 11.1. Now, for an arbitrary index j from the set J, we put

$$A_j=\bigcup_{i\in I_j}X_i.$$

It is easy to verify that the family $(A_j)_{j\in J}$ satisfies all conditions of the lemma, and thus Lemma 11.4 is proved. $\square$

Proof of Theorem 11.2. Let $(A_j)_{j\in J}$ be a family of subsets of the group H whose existence is stated in Lemma 11.4. For arbitrary $f:I\rightarrow\{0;1\}$, denote by $(A^f_j)_{j\in J}$ the family of subsets of the group H defined by

$$j \in J \to A_j^f = \begin{cases} A_j, & \text{if } f(j) = 0, \\ H \setminus A_j, & \text{if } f(j) = 1. \end{cases}$$

Let K_f be the H-invariant σ-ideal generated by the family $(A_j^f)_{j \in J}$. Then it is easy to verify that the ideal K satisfies all conditions of Lemma 11.3. Let us denote by μ_f the H-invariant extension of the Haar measure produced by the ideal K_f. We obtain the family $(\overline{\mu}_f)_{f \in \Phi}$ of H-invariant extensions of the Haar measure λ such that

$$\Phi = \{0;1\}^J.$$

Denote by S the H-invariant σ-algebra of subsets of the group H, generated by the union

$$F(H) \cup \mathcal{B}(H) \cup (\bigcup_{j \in \Phi} \{A_j\}).$$

Also, assume that

$$(\forall f)(f \in \Phi \to \mu_f = \overline{\mu}_f|_S).$$

If we consider the family of H-invariant measures

$$(\mu_t)_{t \in T} = (\mu_f)_{f \in \Phi},$$

we can easily conclude that this family satisfies all conditions of Theorem 11.2, which completes the proof of the theorem. $\square$

It is of interest to consider the question of constructing a maximal (in the sense of cardinality) family of orthogonal invariant extensions of the Haar measure in the locally-compact topological group H for which the condition

$$card(H^{\aleph_0}) > card(H)$$

holds.

In this direction, the following result is of interest.

Theorem 11.3 *Assume that the Generalized Continuum Hypothesis is valid. If an infinite space E satisfies the condition $card(E^{\aleph_0}) > card(E)$, then the maximal cardinality of the orthogonal family of σ-finite measures defined on the space E is equal to $card(E)$.*

Proof. Assume the contrary. Then there exist a σ-algebra S of subsets of the space E and a family $(\mu_t)_{t \in T}$ of orthogonal probability measures on S such that

$$card(T) > card(E).$$

Applying Lemma 11.1, we have

$$E = \sum_{i \in \Omega} E_i,$$

where $card(\Omega) \leq \aleph_0$, and

$$(\forall i)(i \in \Omega \to card(E_i) < card(E)).$$

If $card(E) \leq \aleph_0$, then we can assume that E_i contains only one point of the set E.

For an arbitrary index $t \in T$, there exists an index $i_t \in \Omega$ such that

$$\mu_t^*(E_{i_t}) = \inf\{\mu_t(A) : E_{i_t} \subseteq A \ \& \ A \in S\} > 0.$$

Let us consider the measurable hull B_{i_t} of the set E_{i_t} and define the functional $\overline{\mu}_t$ on the σ-algebra

$$S_0 = \{Y | (\exists (X_i)_{i \in \Omega})((\forall i)(i \in I \rightarrow$$
$$\rightarrow (X_i \in S) \ \& \ (Y = \sum_{i \in \Omega}(E_i \cap X_i))\}$$

by

$$\overline{\mu}_t(\sum_{i \in \Omega}(E_i \cap X_i)) = \frac{1}{\mu_t(B_{i_t})}\mu_t(B_{i_t} \cap X_{i_t}).$$

Let us verify the correctness of this definition.

Assume that

$$\sum_{i \in \Omega}(E_i \cap X_i) = \sum_{i \in \Omega}(E_i \cap Y_i),$$

where

$$(\forall i)(i \in \Omega \rightarrow X_i \in S \ \& \ Y_i \in S).$$

Suppose for a moment that

$$\overline{\mu}_t((X_{i_t} \setminus Y_{i_t}) \cap B_{i_t}) > 0.$$

Then we obtain $\mu_t^*((X_{i_t} \setminus Y_{i_t}) \cap B_{i_t} \cap E_{i_t}) > 0$. On the other hand, we have

$$(X_{i_t} \setminus Y_{i_t}) \cap B_{i_t}) \cap E_{i_t} = \emptyset$$

because $E_{i_t} \cap X_{i_t} = E_{i_t} \cap Y_{i_t}$.

This is the contradiction and the correctness of the definition of the functional $\overline{\mu}_t$ $(t \in T)$ is proved.

It is easy to verify that $\overline{\mu}_t$ is a probability measure for all $t \in T$.

Note that $(\overline{\mu}_t)_{t \in T}$ is an orthogonal family.

Assume

$$K_i = \{t | t \in T \ \& \ \overline{\mu}_t(E_i) = 1\}.$$

It is clear that $T = \cup_{i \in \Omega} K_i$, where

$$(\forall i)(i \in \Omega \rightarrow card(K_i) < card(E)).$$

Let us consider two cases.

I. Let $card(E) \leq \aleph_0$. Then, for an arbitrary index $i \in \Omega$, the set E_i is a subset of the space E with only one point. Since

$$card(T) \leq \sum_{i \in \Omega} card(K_i) \leq card(\Omega) = \min\{card(\Omega), \aleph_0\} \leq card(E),$$

we have a contradiction with the condition

$$card(T) > card(E).$$

$II. If\ card(E) > \aleph_0$, then applying the Generalized Continuum Hypothesis, we have

$$card(T) \le \sum_{i\in\Omega} card(K_i) \le \sum_{i\in\Omega} 2^{card(K_i)} \le$$

$$\le card(\Omega) \times card(E) \le \aleph_0 \times card(E) = card(E),$$

and we also obtain a contradiction with the condition card $(T) >$ card(E). This completes the proof of Theorem 11.3. $\square$

Theorem 11.4 *Let H be a locally compact σ-compact topological group and λ be the Haar measure defined on this group. Then there exists a family $(\mu_t)_{t\in I}$ of orthogonal measures such that:*

1) $card(I) = card(H)$.

2) $(\forall t)(t \in I \to \mu_t$ *is an H-invariant extension of the Haar measure* $\lambda)$.

Proof. Let $(X_i)_{i\in I}$ be the family of subsets of the space H whose existence is formulated in Lemma 11.2. For arbitrary $i \in I$, define the H-invariant measure $\overline{\mu}_i$ by the formula

$$\overline{\mu}_i((X_i \cap A) \cup [(H \setminus X_i) \cap B]) = \lambda(A),$$

where $A \in \mathcal{B}(H), B \in \mathcal{B}(H)$.

If we denote by S_0 an σ-algebra generated by the family of sets $(X_i)_{i\in I} \cup \mathcal{B}(H)$, and consider the family of measures

$$(\mu_i)_{i\in I} = (\overline{\mu}_i|_{S_0})_{i\in I},$$

then Theorem 11.4 will be proved. $\square$

Now, we will consider an application of the method of independent families of sets, in particular, the method of constructing nonelementary nonseparable invariant extensions of the Haar measure for an uncountable σ-compact locally compact topological group H with $card(H^{\aleph_0}) = card(H)$.

We want to recall some definitions from measure theory which will be applied in the sequel.

Let (E,G,S,μ) be an invariant measurable space with invariant measure. An element $X \in S$ is called μ-almost G-invariant if the condition

$$(\forall g)(g \in G \to \mu(g(X) \triangle X) = 0)$$

is fulfilled.

Let (E,G,S,μ) be a space with an invariant measure and X be a μ-almost G-invariant subset of this space. Following [85], the function

$$\mu_X : S \to \overline{R}^+$$

defined by the formula

$$(\forall Z)(Z \in S \rightarrow \mu_X(Z) = \mu(X \cap Z))$$

is called a component of the measure μ associated with the set X.

Analogously, the component μ_X of the measure μ is an elementary component of μ if, for arbitrary $Z \in S$ with $\mu(Z) > 0$, there exists a subsequence $(g_k)_{k \in N}$ of the group G such that

$$\mu(X \setminus \bigcup_{k \in N} g_k(Z)) = 0.$$

A G-invariant measure μ is nonelementary if it does not have any elementary component.

Also note that the function ρ_μ, defined by

$$(\forall X)(\forall Y)(X \in S \ \& \ Y \in S \rightarrow \rho_\mu(X,Y) = \mu(X \triangle Y)),$$

is a quasimetric defined on the class $\mathrm{dom}(\mu) = S$ of all μ-measurable subsets of the base space E;

The pair $(dom(\mu), \rho_\mu)$ is called a metric space associated with the measure μ.

The measure μ is called separable (nonseparable) if the topological weight $a(\mu)$ of the metric space $(dom(\mu), \rho_\mu)$ associated with the measure μ satisfies the condition

$$a(\mu) < \aleph_1 \ (a(\mu) \geq \aleph_1),$$

where $\aleph_1$ denotes the first uncountable cardinal number.

The following lemma is valid.

Lemma 11.5 *Let E be an uncountable base space with $card(E^{\aleph_0}) = card(E)$. Then there exists a nonatomic probability measure $\mathcal{P}$ such that the following conditions hold:*

a) $(\forall X)(X \subseteq E \ \& \ card(X) < card(E) \rightarrow \mathcal{P}(X) = 0)$;

b) the topological weight $a(\mathcal{P})$ of the metric space $(dom(\mathcal{P}), \rho_{\mathcal{P}})$ associated with measure $\mathcal{P}$ is maximal, in particular, is equal to $2^{card(E)}$.

Proof. Denote by S the algebra of parts of the space E generated by the union $\{E, \emptyset\} \cup \{A_i\}_{i \in J}$, where $(A_i)_{i \in J}$ is the family of subsets of the space E constructed in Lemma 11.4.

All elements of the algebra S have the form

$$\bigcup_{\psi \in \{0;1\}^p} ((A_{j_1}^{\psi(1)} \cap \cdots \cap A_{j_p}^{\psi(p)}) \cap Z_\psi),$$

where p is an arbitrary natural number, $\{0;1\}^p$ is the set of all functions from the set $\{1, \cdots, p\}$ into the set $\{0;1\}$, $(A_{j_r})_{1 \leq r \leq p}$ is a sequence of pairwise distinct elements from the family $(A_j)_{j \in J}$ and, finally, for arbitrary $\psi \in \{0;1\}^p$ and $r \in \{1, \cdots, p\}$

$$Z_\psi \in \{\emptyset; E\}, \ A_{j_r}^{\psi(r)} = \begin{cases} A_{j_r}, & \text{if } \psi(r) = 0; \\ H \setminus A_{j_r}, & \text{if } \psi(r) = 1. \end{cases}$$

Using the property of the family $(A_i)_{i \in J}$, from the equality

$$\bigcup_{\psi \in \{0;1\}^p} (A_{j_1}^{\psi(1)} \cap \cdots \cap A_{j_p}^{\psi(p)} \cap Z_\psi) =$$

$$= \bigcup_{\psi \in \{0;1\}^p} (A_{j_1}^{\psi(1)} \cap \cdots \cap A_{j_p}^{\psi(p)} \cap Z_{\psi}^{'}),$$

we have

$$Z_{\psi} \triangle Z'_{\psi} = \emptyset$$

for all $\psi \in \{0,1\}^p$.

This phenomena allows us to define correctly a measure P by the following formula

$$P(\bigcup_{\psi \in \{0;1\}^p} (A_{j_1}^{\psi(1)} \cap \cdots \cap A_{j_p}^{\psi(p)} \cap Z_{\psi})) = \frac{1}{2^p} \sum_{\psi \in \{0;1\}^p} \lambda(Z_{\psi}),$$

where

$$X \in \{E;\emptyset\} \to \lambda(X) = \begin{cases} 1, & \text{if } X = E, \\ 0, & \text{if } X = \emptyset. \end{cases}$$

It is easy to verify that the functional P is a finitely-additive extension of the measure λ. Note that P is a also finitely-additive function and, therefore, is a measure defined on algebra S. It is sufficient to show that, for an arbitrary positive real number ε with $0 < \varepsilon < 1$ and for an arbitrary sequence $(Y_k)_{k \in N}$ such that

$$a)\ (\forall k)(k \in N \to Y_k \in S),$$

$$b)\ (\forall k)(k \in N \to Y_{k+1} \subseteq Y_k),$$

$$c)\ (\forall k)(k \in N \to P(Y_k) \geq \varepsilon),$$

$$d)\ P(Y_0) < +\infty,$$

the condition

$$\bigcap_{k \in N} Y_k \neq \emptyset$$

holds.

Note that an arbitrary set Y_k $(k \in N)$ can be represented by the formula

$$Y_k = \bigcup_{\psi \in \{0;1\}^{p_k}} (A_{j_1}^{\psi(1)} \cap \cdots \cap A_{j_{p_k}}^{\psi(p_k)} \cap Z_{\psi}^{(k)}).$$

Without loss of generality, we can assume that the sequence of classes

$$(\{(A_{j_r} \,:\, 1 \leq r \leq p_k\})_{k \in N}$$

is increased by inclusion.

By using the validity of the condition

$$\frac{1}{2^{p_k}} \sum_{\psi \in \{0;1\}^{p_k}} \lambda(Z_{\psi}^{(k)}) \geq \varepsilon,$$

we can deduce that such a set exists for an arbitrary natural number k. We can easily conclude that there exists a sequence $(Z^{(k)}_{\psi_k})_{k\in N}$ such that:

$a)$ $(\forall k)(k \in N \rightarrow \lambda(Z^{(k+1)}_{\psi_{k+1}} \setminus Z^{(k)}_{\psi_k}) = 0 \ \& \ \lambda(Z^{(k)}_{\psi_k}) \geq \varepsilon)$;

$b)$ $(\forall k)(k \in N \rightarrow \psi_{k+1}(i) = \psi_k(i)$ for $1 \leq i \leq p_k)$.

Therefore,

$$(\forall k)(k \in N \rightarrow Z^{(k)}_{\psi_k} = E)$$

and we obtain

$$\bigcap_{k\in N}(A^{\psi_k(1)}_{j_1} \cap \cdots \cap A^{\psi_k(p_k)}_{j_{p_k}} \cap Z^{(k)}_{\psi_k}) \neq \emptyset.$$

Hence it follows that the condition

$$\bigcap_{k\in N} Y_k \neq \emptyset$$

is fulfilled.

Thus, P is a measure defined on the algebra S. According to the Carathéodory theorem, this measure can be extended in a unique way on the σ-algebra generated by

$$\{E;\emptyset\} \cup \{A_j \ : \ j \in J\}.$$

Let us denote by $\overline{P}$ the extended measure. Finally, we denote by $\mathcal{P}$ the functional defined by

$$(\forall X)(\forall X')(\forall X'')(X \in dom(\overline{P}) \ \& \ X' \subseteq E \ \& \ card(X') <$$

$$< card(E) \ \& \ X'' \subseteq E \ \& \ card(X'') < card(E) \rightarrow$$

$$\rightarrow \mathcal{P}((X \setminus X') \cup X'') = \overline{P}(X)).$$

By using the condition $card(E^{\aleph_0}) = card(E)$, we conclude that the functional $\mathcal{P}$ is the extension of the probability measure P.

Let us observe that $\alpha(\mathcal{P}) = 2^{card(E)}$. If we consider the family $(A_j)_{j\in J}$ of sets, we find that

$$(\forall j)(\forall j')(j \in J \ \& \ j' \in J \ \& \ j \neq j' \rightarrow \rho_{\mathcal{P}}(A_j, A_{j'}) = \mathcal{P}(A_j \triangle A_{j'}) = \frac{1}{2}).$$

This completes the proof of Lemma 11.5. □

The following lemma is valid.

Lemma 11.6 *Let H be a σ-compact locally compact topological group with $card(H^{\aleph_0}) = card(H)$. Let*

$$\mathcal{F}(H) = \{Y | Y \subseteq H \ \& \ card(Y) < card(H)\},$$

$$\mathcal{F}(H \times H) = \{Z | Z \subseteq H \times H \ \& \ card(Z) < card(H)\}.$$

Let μ be a probability measure defined on the space H such that

$$(\forall Y)(Y \in \mathcal{F}(H) \rightarrow \mu(Y) = 0).$$

Let λ be the Haar measure defined on the topological group H. Let us denote by $\overline{\lambda}$ the extension of λ defined by

$$(\forall Y)(\forall Y_1)(\forall Y_2)(Y \in \mathcal{B}(H)\ \&\ Y_1 \in \mathcal{F}(H)\ \&\ Y_2 \in \mathcal{F}(H) \rightarrow$$

$$\overline{\lambda}((Y \setminus Y_1) \cup Y_2) = \lambda(Y)).$$

Then the functional $\widehat{(\mu \times \overline{\lambda})}$ defined by

$$(\forall X)(\forall X')(\forall X'')(X \in dom(\mu \times \overline{\lambda})\ \&\ X' \in \mathcal{F}(H \times H)\ \&$$

$$X'' \in \mathcal{F}(H \times H) \rightarrow \widehat{(\mu \times \overline{\lambda})}((X \setminus X') \cup X'') = (\mu \times \overline{\lambda})(X))$$

is the $\{e\} \times H$-invariant measure, where $\{e\}$ is the unit element of the group H.

The proof of Lemma 11.6 is not difficult and can be obtained by the scheme considered in [85].

Remark 11.2 It should be noted that the equality

$$card(H)^{\aleph_0} = card(H)$$

is not a significant restriction for a nondiscrete σ-compact locally compact topological group H. Indeed, it is not hard to prove that any such group is an isodyne topological space of the second Baire category. In this context, we apply the result obtained in [90], which states that, under the Generalized Continuum Hypothesis, any second category isodyne space H satisfies above-mentioned equality. This shows that GCH automatically implies the validity of the basic assumption $card(H)^{\aleph_0} = card(H)$ which participates in the formulations of all main statements of this paper. Note also that it is well known in the theory of general topology that under the Generalized Continuum Hypothesis, for any nondiscrete σ-compact locally compact topological group H there exists a such infinite cardinal number J that the equality $\text{card}(H) = 2^J$ holds. This equality automatically implies the validity of above-mentioned equality.

We have the following statement.

Theorem 11.5 *Let $(x_i)_{i \in I}$ be a family of all elements of some uncountable locally compact σ-compact topological group H with $card\,(H^{\aleph_0}) = card(H)$. Let $(X_i)_{i \in I}$ be the family of subsets of H constructed in Lemma 11.2.*

Define the functional ψ by

$$(\forall x)(x \in H \rightarrow \psi(x) = (x_i, x)),$$

where i is a unique index from the parametric set I for which $x \in X_i$.

We conclude that the functional λ_μ defined by

$$(\forall X)(X \in dom\widehat{(\mu \times \overline{\lambda})} \rightarrow \lambda_\mu(\psi^{-1}(X)) = \widehat{(\mu \times \overline{\lambda})}(X))$$

is the H-invariant extension of the Haar measure λ, where the extension λ_μ is nonelementary if and only if the measure μ is nonatomic.

Proof. We must show that the domain of the definition of λ_μ is an H-invariant σ-algebra. This means that

$$(\forall X)(\forall g)(X \in dom(\widehat{\mu \times \overline{\lambda}}) \ \&\ g \in H \to$$

$$(\exists Y)\ (Y \in dom(\widehat{\mu \times \overline{\lambda}}) \to g(\psi^{-1}(X)) = \psi^{-1}(Y)).$$

We also have

$$(\forall X_1)(\forall X_2)(\forall X')(\forall X'')(\forall g)(X_1 \in \mathrm{dom}(\overline{\mu}) \ \&\ X_2 \in \mathrm{dom}(\overline{\lambda}) \ \&\ X'$$

$$\in \mathcal{F}(H \times H) \ \&\ X'' \in \mathcal{F}(H \times H) \ \&\ g \in H \to (\exists Y_1)(\exists Y_2)(\exists Y')$$

$$(\exists Y'')(Y_1 \in \mathrm{dom}(\overline{\mu}) \ \&\ Y_2 \in \mathrm{dom}(\overline{\lambda}) \ \&\ Y' \in \mathcal{F}(H \times H) \ \&\ Y'' \in$$

$$\mathcal{F}(H \times H) \to g(\psi^{-1}(((X_1 \times X_2) \setminus X') \cup X'')) = \psi^{-1}(((Y_1 \times Y_2) \setminus Y') \cup Y'')))).$$

Indeed,

$$g(\psi^{-1}(((X_1 \times X_2) \setminus X') \cup X'')) = g\{[\textstyle\bigcup_{i \in X_1}(X_i \cap X_2) \setminus \psi^{-1}(X')] \cap$$

$$\psi^{-1}(X'')\} = [[\textstyle\bigcup_{i \in X_1}(X_i \cap g(X_2))] \setminus [g(\psi^{-1}(X'))] \cup [g(\psi^{-1}(X''))] =$$

$$= [\textstyle\bigcup_{i \in X_1}(X_i \cap g(X_2)) \setminus \psi^{-1}(\psi(g(\psi^{-1}(X))))] \cup \psi^{-1}(\psi(g(\psi^{-1}(X'')))) =$$

$$= \psi^{-1}((X_1 \times g(X_2)) \setminus \psi(g(\psi^{-1}(X'))) \cup \psi(g(\psi^{-1}(X'')))).$$

Let us verify that the measure λ_μ is an H-invariant extension of the measure $\overline{\lambda}$. This means that the condition

$$(\forall Y_1)(\forall Y_2)(\forall g)(Y_1 \in \mathrm{dom}(\overline{\mu}) \ \&\ Y_2 \in \mathrm{dom}(\overline{\lambda}) \ \&\ g \in H \to$$

$$\to \lambda_\mu(\psi^{-1}(Y_1 \times Y_2)) = \lambda_\mu(g(\psi^{-1}(Y_1 \times Y_2))))$$

holds.

Indeed, on the one hand, we have

$$g(\psi^{-1}(Y_1 \times Y_2)) = g(\psi^{-1}(\textstyle\bigcup_{i \in J}((Y_1 \times Y_2) \cap \widetilde{X}_i))) = g(\bigcup_{x_i \in Y_1}(Y_2 \cap X_i)) =$$

$$= g(Y_2 \cap (\textstyle\bigcup_{x_i \in Y_1} X_i)) = [g(Y_2) \cap (\bigcup_{x_i \in Y_1} X_i)] \triangle Y',$$

where $\widetilde{X}_i = \{x_i\} \times X_i$ and $card(Y') < card(H)$.

On the other hand, we have

$$\psi^{-1}(\widetilde{g}(Y_1 \times Y_2)) = \psi^{-1}(Y_1 \times g(Y_2)) = \psi^{-1}((Y_1 \times g(Y_2)) \cap \textstyle\bigcup_{i \in I} \widetilde{X}_i) =$$

$$= \psi^{-1}(\textstyle\bigcup_{x_i \in Y_1}(\{x_i\} \times g(Y_2)) \cap \widetilde{X}_i) = \bigcup_{x_i \in Y_1}(g(Y_2) \cap X_i) =$$

$$= g(Y_2) \cap (\textstyle\bigcup_{x_i \in Y_1} X_i).$$

Thus, we get
$$\lambda_\mu(g(\psi^{-1}(Y_1\times Y_2))\triangle\psi^{-1}(\widetilde{g}(Y_1\times Y_2)))=0.$$

From the equality

$$\lambda_\mu(\psi^{-1}(\widetilde{g}(Y_1\times Y_2)))=\lambda_\mu(\psi^{-1}(Y_1\times g(Y_2)))=\lambda_\mu(g(\psi^{-1}(Y_1\times Y_2)))$$

we have
$$\lambda_\mu(\psi^{-1}(Y_1\times Y_2))=\lambda_\mu(g(\psi^{-1}(Y_1\times Y_2))).$$

Now, we must prove that the measure λ_μ is nonelementary if and only if the measure μ is nonatomic.

Let μ be a nonatomic measure.

Suppose the contrary and let, for some $X\in dom(\lambda_\mu)$, a measure $\lambda_\mu|_X$ be an elementary component of the measure λ_μ. This means that the following conditions hold:

1) X is a λ_μ-almost H-invariant subset of the space H;

2) $(\forall B)(B\in \mathrm{dom}(\lambda_\mu)\rightarrow(\lambda_\mu|_X)(B)=\lambda_\mu(B\cap X)$;

3) $(\forall Y)(Y\in \mathrm{dom}(\lambda_\mu)\ \&\ (\lambda_\mu|_X)(Y)>0\rightarrow(\exists(g_k)_{k\in N})$
$((\forall k)(k\in N\rightarrow g_k\in H)\ \&\ (\lambda_\mu|_X)(X\setminus\bigcup_{k\in N}g_k(Y))=0)))$.

Denote by Z a $\widehat{\mu\times\overline{\lambda}}$-measurable subset of $H\times H$ for which the equality $\psi^{-1}(Z)=X$ holds.

Let A_1 and A_2 be disjoint elements of the σ-algebra $\mathrm{dom}(\mu)$, for which the conditions

$$(\widehat{\mu\times\overline{\lambda}})(Z\cap(A_1\times H))>0\ \mathit{and}\ (\widehat{\mu\times\overline{\lambda}})(Z\cap(A_2\times H))>0$$

hold.

Let us consider a λ_μ-measurable subset

$$Y=\psi^{-1}((A_1\times H)\cap Z).$$

Note that, for an arbitrary countable family $(g_k)_{k\in N}$ of elements from the group H, we have
$$\lambda_\mu(X\setminus\bigcup_{k\in N}g_k(Y))>0$$
because $\psi^{-1}(A_2\times H)\subseteq X\setminus\bigcup_{k\in N}g_k(Y)$ and $\lambda_\mu(\psi^{-1}(A_2\times H))>0$.

We have obtained a contradiction, and the nonelementarity of the measure λ_μ is proved.

Now, suppose λ_μ is a nonelementary measure and let μ be an atomic measure. The latter means that there exists an atom A of the measure μ. Then the subset $X=\psi^{-1}(A\times H)$ is a λ_μ-almost G-invariant subset of H and the measure $(\lambda_\mu|_X)(B)=\lambda_\mu(X\cap B)(B\in dom(\lambda_\mu))$ is an elementary component of the measure λ_μ.

The latter statement is a contradiction and Theorem 11.5 is proved. $\square$

By using Theorem 11.5 and Lemma 11.5, we can prove the following theorem.

Theorem 11.6 *Let H be an uncountable locally compact σ-compact topological group with $card(H^{\aleph_0})=card(H)$. Let λ be the Haar measure defined on the topological group H. Then there exists an H-invariant extension $\widetilde{\lambda}$ of the measure λ such that:*

1) *the measure $\widetilde{\lambda}$ is nonelementary;*

2) *the topological weight of the metric space* $(dom(\widetilde{\lambda}), \rho_{\widetilde{\lambda}})$ *is maximal; in particular, it is equal to* $2^{card(H)}$.

The proof of Theorem 11.6 can be obviously obtained if we denote by $\widetilde{\lambda}$ the measure $\lambda_{\mathcal{P}}$, where $\mathcal{P}$ is the measure constructed in Lemma 11.5.

Remark 11.3 The construction of an H-invariant nonelementary extension of the Haar measure cannot be used in the case of a locally compact σ-compact topological group with $card(H^{\aleph_0}) \neq card(H)$ because if the General Continuum Hypothesis is valid, then the group H can be covered by a countable family of H-absolutely negligible subsets of H whose every element has cardinality less than the cardinality of the group H.

The following theorem is valid.

Theorem 11.7 *Let H be an uncountable locally compact σ-compact topological group with $card(H^{\aleph_0}) = card(H)$. Let λ be the Haar measure defined on the topological group H. Then there exists a maximal (in the sense of cardinality) orthogonal family $(\lambda_t)_{t\in T}$ of H-invariant nonelementary extensions of the Haar measure with $card(T) = 2^{2^{card(H)}}$ such that :*

1) $(\forall i)(\forall j)(i \in T \ \& \ j \in T \rightarrow dom(\lambda_i) = dom(\lambda_j))$;
2) $(\forall i)(i \in T \rightarrow \alpha(\lambda_i)$ *is maximal* & $\alpha(\lambda_i) = 2^{card(H)})$.

Proof. Let $(\widetilde{\mu}_t)_{t\in T}$ be an orthogonal family of probability measures such that:
$A)$ $(\forall t)(t \in T \rightarrow \widetilde{\mu}_t$ is a nonatomic probability measure defined on $H)$;
$B)$ $(\forall t_1)(\forall t_2)(t_1 \in T \ \& \ t_2 \in T \rightarrow \mathrm{dom}(\widetilde{\mu}_{t_1}) = \mathrm{dom}(\widetilde{\mu}_{t_2}))$;
$C)$ $(\forall t)(\forall X)(t \in T \ \& \ X \subseteq H \ \& \ card(X) < card(H) \rightarrow \widetilde{\mu}(X) = 0)$;
$E)$ $card(T) = 2^{2^{card(H)}}$.

By Lemma 11.5, we can construct the measure $\mathcal{P}$ defined on the set $E = H$. Denote by $(\widetilde{\mu}_t \times \mathcal{P})^{\maltese}(t \in T)$ the extension of the probability measure $\widetilde{\mu}_t \times \mathcal{P}$ obtained by means of elements from the class $\mathcal{F}(H \times H)$, where

$$\mathcal{F}(H \times H) = \{Y | Y \subseteq H \times H \ \& \ card(Y) < card(H)\}.$$

Let f be a one-to-one correspondence between the sets $H \times H$ and H. Let us consider the family $(f((\widetilde{\mu}_t \times \mathcal{P})^{\maltese}))_{t\in T}$ of measures defined by

$$(\forall i)(\forall X)(i \in T \ \& \ X \in \mathrm{dom}((\widetilde{\mu}_t \times \mathcal{P})^{\maltese}) \rightarrow$$

$$\rightarrow f((\widetilde{\mu}_t \times \mathcal{P})^{\maltese})(f(X)) = (\widetilde{\mu}_i \times \mathcal{P})(X)).$$

Using the method of construction of nonelementary extensions of the Haar measure considered in Theorem 11.5, we can put

$$(\forall t)(t \in T \rightarrow \lambda_t = \lambda_{f((\widetilde{\mu}_t \times \mathcal{P})^{\maltese})}).$$

It is easy to verify that the family $(\lambda_t)_{t\in T}$ of measures satisfies all the conditions of Theorem 11.7. The theorem is proved. $\square$

Let us recall the following well-known notion of a density point.

Let μ be some extension of the n-dimensional classical Lebesgue measure l_n and X be some μ-measurable subset of $\mathbf{R}^n$. A point $x \in \mathbf{R}^n$ is called a density point of X with respect to the Vitali standard system generated by the family of all n-dimensional cubes of the space $\mathbf{R}^n$ if

$$\lim_{h \to 0^+} \frac{\mu(X \cap [x-h; x+h]^n)}{2^n \cdot h^n} = 1.$$

Recall that an arbitrary D_n-invariant extension of the measure l_n, where D_n denotes the group of all isometric transformations, is called a D_n-measure.

Note that the method considered above gives a solution of one problem formulated by A.B.Kharazishvili [85] (see p. 200, problem $\mathcal{N}9$).

Problem 11.1 *Does there exist a D_n-measure μ in the Euclidean space $\mathbf{R}^n$ such that some μ-measurable subset has only one point of density with respect to the Vitali standard system generated by the family of all n-dimensional cubes of the space $\mathbf{R}^n$?*

We are going to discuss this problem for the 1-dimensional classical Lebesgue measure l_1.

Let λ_1 be the standard Lebesgue measure defined on the space $[0;1]$. It is easy to verify that $\lambda_1 \times l_1$ is a $\widetilde{D}_1$-invariant measure, where $\widetilde{D}_1$ is the group of transformations of the space $[0;1] \times \mathbf{R}$ defined by the formula

$$\widetilde{D_1} = \{I\} \times D_1,$$

where $\{I\}$ is the identity transformation of the interval $[0;1]$, D_1 is the group of all isometric transformations of the real line $\mathbf{R}$.

Let

$$\mathcal{F}([0;1] \times \mathbf{R}) = \{Y | Y \subseteq [0;1] \times \mathbf{R} \; \& \; card(Y) < \mathbf{c}\}.$$

It is easy to verify that the functional λ defined by

$$(\forall X')(\forall X'')(\forall X)(X' \in \mathcal{F}([0;1] \times R) \; \& \; X'' \in \mathcal{F}([0;1] \times R) \; \&$$

$$X \in dom(\lambda_1 \times l_1) \to \lambda((X \setminus X') \cup X'') = (\lambda_1 \times l_1)(X)),$$

is a $\widetilde{D}_1$-invariant extension of the measure $\lambda_1 \times l_1$.

We will need the following lemma.

Lemma 11.7 *Let $n \geq 1$ and let $\mathbf{c}$ denote the cardinality of the continuum. Then there exists a family $(X_i)_{i \in [0;1]}$ of subsets of the Euclidean space $\mathbf{R}^n$ such that:*

1) $(\forall i)(\forall i')(i \in [0;1] \; \& \; i' \in [0;1] \; \& \; i \neq i' \to X_i \cap X_{i'} = \emptyset)$;

2) $\bigcup_{i \in [0;1]} X_i = \mathbf{R}^n$;

3) $(\forall i)(\forall F)(i \in [0;1] \; \& \; F$ *is a closed subset of the space* $\mathbf{R}^n$ *with a positive Lebesgue measure* $\to card(X_i \cap F) = \mathbf{c})$;

4) $(\forall I')(\forall g)(I' \subseteq [0;1] \; \& \; g \in D_n \to card(g(\bigcup_{i \in I'} X_i) \triangle (\bigcup_{i \in I'} X_i)) < \mathbf{c})$.

The proof of Lemma 11.7 can be found in [85].

Let $n = 1$ and $(X_i)_{i \in [0;1]}$ be the family of subsets of the space $\mathbf{R}$ which is considered in Lemma 11.7. Define the functional $\psi : \mathbf{R} \to [0;1] \times \mathbf{R}$ by

$$(\forall x)(x \in \mathbf{R} \to \psi(x) = (i, x)),$$

where i is a unique index from the interval $[0;1]$ for which the condition $x \in X_i$ holds.

The following theorem is valid.

Theorem 11.8 *The functional μ, defined by*

$$(\forall X)(X \in dom(\lambda) \to \mu(\psi^{-1}(X)) = \lambda(X)),$$

is a D_1-invariant nonelementary extension of the Lebesgue measure l_1.

Proof. First, let us show the correctness of the definition of the functional μ. Assume the contrary and suppose $A \in dom(\lambda)$ and $B \in dom(\lambda)$ are subsets of the space $[0;1] \times \mathbf{R}$ such that the conditions

$$1)\ \psi^{-1}(A) = \psi^{-1}(B),$$

$$2)\ \lambda(A \triangle B) > 0$$

hold.

Let $\lambda(A \setminus B) > 0$. By Lemma 11.7, we obtain the existence of an index $i_0 \in [0;1]$ and a point $x_0 \in \mathbf{R}$ such that

$$\{i_0\} \times \{x_{i_0}\} \in (A \setminus B) \cap (\{i_0\} \times X_{i_0}).$$

This means that

$$x_{i_0} \in \psi^{-1}(A) \setminus \psi^{-1}(B).$$

We have obtained a contradiction with the condition $\psi^{-1}(A) = \psi^{-1}(B)$ and the correctness of the definition of the functional μ is proved.

Let us verify that μ is a D_1-invariant functional. It is sufficient to verify the validity of the condition

$$(\forall a)(\forall b)(\forall c)(\forall d)(\forall g)(0 \leq a < b \leq 1\ \&\ -\infty < c < d < \infty\ \&\ g \in D_1 \to$$

$$\to \mu(g(\psi^{-1}([a;b] \times [c;d]))) = \mu(\psi^{-1}([a;b] \times [c;d]))).$$

Note that

$$\mu(g(\psi^{-1}([a;b] \times [c;d]))) = \mu(\psi^{-1}([a;b] \times g([c;d]))).$$

Indeed, by Lemma 11.7, on the one hand,

$$g(\psi^{-1}([a;b] \times [c;d])) = g((\bigcup_{i \in [a;b]} X_i) \cap [c;d]) =$$

$$= ((\bigcup_{i \in [a;b]} X_i) \cap g([c;d])) \setminus X' \cup X'',$$

where X' and X'' are some subsets of the space $\mathbf{R}$ whose cardinalities are strictly less than $\mathbf{c}$, and, on the other hand,

$$\psi^{-1}([a;b] \times [c;d]) = (\bigcup_{i \in [a;b]} X_i) \cap g([c;d]).$$

By the inclusion

$$g(\psi^{-1}([a;b]\times[c;d]))\triangle\psi^{-1}([a;b]\times g([c;d]))\subseteq X'\cup X'',$$

we have

$$\mu(g(\psi^{-1}([a;b]\times[c;d])))=\mu(\psi^{-1}([a;b]\times g([c;d]))).$$

Finally, by the $\widetilde{D}_1$-invariance of the measure λ we have

$$\mu(g(\psi^{-1}([a;b]\times[c;d])))=\mu(\psi^{-1}([a;b]\times g([c;d])))=$$

$$=\lambda([a;b]\times g([c;d]))=\lambda([a;b]\times[c;d])=\mu(\psi^{-1}([a;b]\times[c;d])).$$

The nonelementarity of the measure μ can be established by the scheme considered in the proof of Theorem 11.5. Theorem 11.8 is proved. □

The basic result is formulated by the following theorem.

Theorem 11.9 *There exists a μ-measurable subset of the space* **R** *which has only one density point with respect to the Vitali standard system generated by the family of all open intervals of* **R**.

Proof. Let us consider a μ-measurable set $\psi^{-1}(K)$, where K is the closed square whose vertices are at the points $(0;0),(\frac{1}{2},\frac{1}{2}),(1;0),(\frac{1}{2},\frac{-1}{2})$. Let $x\in\mathbf{R}$ and $\{(a_k,b_k)\}_{k\in N}$ be an arbitrary sequence of open intervals, fundamental at the point x. By using the construction of the measure μ, we have

$$\lim_{k\to\infty}\frac{\mu(\psi^{-1}(K)\cap(a_k;b_k))}{\mu((a_k;b_k))}=\begin{cases}1-2|x|, & |x|<\frac{1}{2}\\ 0, & |x|\geq\frac{1}{2}\end{cases}.$$

In particular, this means that the set $\psi^{-1}(K)$ has only one density point $(x=0)$ with respect to the standard Vitali system generated by the family of all open intervals of the real line **R**. □

Remark 11.4 Applying the scheme proposed in theorems 11.8 and 11.9, we can construct a D_n-invariant nonelementary extension μ_n of the classical Lebesgue measure l_n in the Euclidean space $\mathbf{R}^n$, for $n\geq 2$, such that some μ_n-measurable set would have only one density point with respect to the standard Vitali system generated by the family of all n-dimensional cubes of the space $\mathbf{R}^n$, and the topological weight of the metric space $(\mathrm{dom}(\mu_n),\rho_{\mu_n})$ would be maximal, in particular, be equal to $2^{\mathbf{c}}$.

Chapter 12

Separated Families of Probability Measures

Let (E,S) be a measurable space.

Definition 12.1 A family of probability measures $(\mu_i)_{i\in I}$ defined on a measurable space (E,S) is called weakly separated if there exists a family $(X_i)_{i\in I}$ of measurable subsets of E, such that

$$(\forall i)(i \in I \;\&\; j \in I \rightarrow \mu_i(X_j) = \delta(i,j)),$$

where $\delta(i,j)$ denotes Kronecker's function defined on the Cartesian square I^2 of the set I.

Definition 12.2 A family of probability measures $(\mu_i)_{i\in I}$ defined on a measurable space (E,S) is called strictly separated if there exists a disjoint family $(X_i)_{i\in I}$ of measurable subsets of the space E, such that

$$(\forall i)(i \in I \rightarrow \mu_i(X_i) = 1).$$

It is clear that an arbitrary strictly separated family $(\mu_i)_{i\in I}$ of probability measures is weakly separated.

In connection with the definitions above, see [73] where the structure of weakly separated and strictly separated families of probability measures is investigated.

In the general theory of statistical decisions there often arises a question of transition from a weakly separated family of probability measures to the corresponding strictly separated family. In this context, the following result is of particular interest.

Theorem 12.1 *In the system of axioms (ZFC) the following three conditions are equivalent:*

1) The Continuum Hypothesis ($\mathbf{c} = 2^{\aleph_0} = \aleph_1$);

2) for an arbitrary probability space (E,S,μ), the μ-measure of the union of any family $(E_i)_{i\in I}$ of μ-measure zero subsets, such that $card(I) < c$, is equal to zero;

3) an arbitrary weakly separated family of probability measures, of cardinality continuum, is strictly separated.

Proof.

$1) \to 2)$. Let (E,S,μ) be an arbitrary probability space and let $(E_i)_{i\in I}$ be a family of μ-measure zero subsets of E, such that $\mathrm{card}(I) < c$. Applying condition $1)$, we have $\mathrm{card}(I) \leq \omega$, where ω denotes the cardinality of the set of all natural numbers. Finally, applying semiadditivity of the measure μ, we obtain

$$\mu(\cup_{i\in I}E_i) \leq \sum_{i\in I}\mu(E_i) = 0.$$

The implication $1) \to 2)$ is thus proved.

$2) \to 3)$. Let ω_ϕ be the first ordinal number of cardinality of the continuum, let $(\mu_\xi)_{\xi\prec\omega_\phi}$ be a family of probability measures defined on a measurable space (E,S) and suppose that there exists a family $(X_\xi)_{\xi\prec\omega_\phi}$ of measurable subsets of E such that

$$(\forall\xi)(\forall\tau)(\xi \prec \omega_\phi \ \&\ \tau \prec \omega_\phi \to \mu_\xi(X_\tau) = \delta(\xi,\tau)),$$

where $\delta(\xi,\tau)$ denotes Kronecker's function on the Cartesian square

$$[0;\omega_\phi[\times[0;\omega_\phi[$$

of the set $[0;\omega_\phi[$.

Let

$$(\forall\xi)(\xi \prec \omega_\phi \to Y_\xi = X_\xi \setminus \bigcup_{\tau\prec\xi} X_\tau).$$

Utilizing the condition $2)$, we conclude that $(Y_\xi)_{\xi\prec\omega_\phi}$ is a disjoint family of measurable subsets of the space E, such that

$$(\forall\xi)(\xi \prec \omega_\phi \to \mu_\xi(Y_\xi) = 1).$$

This means that the implication $2) \to 3)$ is proved.

$3) \to 1)$. For arbitrary $x \in]0;1[$, define the σ-algebra B_x of subsets of the space $\triangle_2 =]0;1[\times]0;1[$ by

$$B_x = \{\ Y | Y \subseteq \triangle_2 \ \&\ (card(Y \cap (\{\ x\ \} \times]0;1[)) \leq \aleph_0) \vee$$

$$(card((\{\ x\ \} \times]0;1[) \setminus Y) \leq \aleph_0)\ \}\ .$$

For arbitrary $x \in]1;2[$, denote by B_x the σ-algebra of subsets of the space $\triangle_2$ defined by

$$B_x = \{\ Y | Y \subseteq \triangle_2 \ \&\ (card(Y \cap (]0;1[\times \{\ x-1\ \})) \leq \aleph_0) \vee$$

$$(card((]0;1[\times \{\ x-1\ \}) \setminus Y) \leq \aleph_0)\ \}\ .$$

Let us put

$$S = \bigcap_{x \in]0;1[\cup]1;2[} B_x.$$

It is clear that each element of the families

$$(\ \{\ x\ \} \times]0;1[)_{x \in]0;1[} \text{ and } (]0;1[\times \{\ x-1\ \}\ )_{x \in]1;2[}$$

belongs to the σ-algebra S.

Define the family $(\mu_t)_{t \in]0;1[\cup]1;2[}$ of probability measures by

$$(\forall t)(t \in]0;1[\rightarrow \mu_t(Z) = \begin{cases} 1, & \text{if } card((\ \{\ t\ \} \times]0;1[) \setminus Z) \leq \aleph_0, \\ 0, & \text{if } card((\ \{\ t\ \} \times]0;1[) \cap Z) \leq \aleph_0 \end{cases}),$$

$$(\forall t)(t \in]1;2[\rightarrow \mu_t(Z) = \begin{cases} 1, & \text{if } card((]0;1[\times \{\ t-1\ \}\ ) \setminus Z) \leq \aleph_0, \\ 0, & \text{if } card((]0;1[\times \{\ t-1\ \}\ ) \cap Z) \leq \aleph_0 \end{cases})$$

for $Z \in S$.

Let us consider the family $(X_t)_{t \in]0;1[\cup]1;2[}$ of measurable subsets of the space $\triangle_2$, where

$$(\forall t)(t \in]0;1[\cup]1;2[\rightarrow X_t = \begin{cases} \{\ t\ \} \times]0;1[, & \text{if } t < 1, \\ ]0;1[\times \{\ t-1\ \}, & \text{if } t > 1 \end{cases}).$$

It is clear that the family $(\mu_t)_{t \in]0;1[\cup]1;2[}$ of probability measures is weakly separated, because of

$$(\forall t_1)(\forall t_2)((t_1,t_2) \in (]0;1[\cup]1;2[)^2 \rightarrow \mu_{t_1}(X_{t_2}) = \delta(t_1,t_2)),$$

where $\delta(.,.)$ denotes Kronecker's function defined on the Cartesian square

$$(]0;1[\cup]1;2[)^2$$

of the set $]0;1[\cup]1;2[$.

From condition 3) we have that the family $(\mu_t)_{t \in]0;1[\cup]1;2[}$ of probability measures is strictly separated. This means that there exists a family of disjoint measurable subsets $(Y_t)_{t \in]0;1[\cup]1;2[}$ such that

$$(\forall t)(t \in]0;1[\cup]1;2[\rightarrow \mu_t(Y_t) = 1).$$

We may assume, without loss of generality, that $Y_t \subseteq X_t$ for all $t \in]0;1[\cup]1;2[$. Let us consider the sets $A = \bigcup_{t \in]0;1[} Y_t$ and $B = \bigcup_{t \in]1;2[} Y_t$.

It is clear that A and B do not have common points. On the other hand, we can write

$$(\forall x)(x \in]0;1[\rightarrow card((\ \{\ x\ \} \times]0;1[) \cap B) \leq \aleph_0\ \&$$

$$\&\ \text{card}((]0;1[\times \{\ x\ \}\ ) \cap A) \leq \aleph_0).$$

Denote by $(C_\xi)_{\xi \prec \omega_1}$ some injective transfinite sequence of horizontal segments of the space $\triangle_2$. It is clear that

$$card(A \cap (\bigcup_{\xi \prec \omega_1} C_\xi)) \leq \aleph_0 \times \aleph_1 = \aleph_1.$$

We have to prove that the orthogonal projection of the set

$$A \cap (\bigcup_{\xi \prec \omega_1} C_\xi)$$

on the interval $]0;1[\times \{ 0 \}$ coincides with this interval.

Indeed, let a be an arbitrary vertical segment of the space $\triangle_2$. Since,

$$card(B \cap a) \leq \aleph_0,$$

there exists an ordinal index $\xi_0 \prec \omega_1$ such that the point of the intersection of C_{ξ_0} and a belongs to the set A. This means that the set $A \cap (\bigcup_{\xi \prec \omega_1} C_\xi)$ is projected on the whole interval $]0;1[\times \{ 0 \}$ and,therefore,

$$2^{\aleph_0} \leq \aleph_1. \qquad \square$$

Remark 12.1 Note that the implication $1) \rightarrow 3)$ was obtained in [73]. The validity of the implication $3) \rightarrow 1)$ was established in [133].

Remark 12.2 M.Coldstern (5.08.2002) offers a different proof of the equivalence of the conditions 1) and 2). His proof is based on the following fact:

Fact A: There is a measure space and a family of $\aleph_1$-many measure zero sets whose union is not measure zero, and not even measurable.

Notice that Fact **A** is true in the usual axiomatic set theory (e.g.,in ZFC).

One proof of Fact **A** reads as follows:

Take any uncountable set X. Consider the σ-algebra of those subsets of X which are either at most countable or whose complement is at most countable. Define the measure μ by letting $\mu(C) = 0$ and $\mu(X \setminus C) = 1$ whenever C is countable. This is a complete measure and serves as an example for Fact **A**.

Here is the second example (proposed by the same author) with an incomplete measure.

Consider the σ-algebra of Borel sets equipped with the Lebesgue measure.

Then there is a family of $\aleph_1$-many measure zero sets whose union is not measurable. This example can be found in [44](see Volume 5, Exercise 511Xj).

Remark 12.3 In the system of axioms $(ZFC)\&(\neg CH)\&(MA)$ the family of probability measures $(\mu_t)_{t\in]0;1[\cup]1;2[}$ considered in Theorem 12.1, is an example of a weakly separated family of probability measures, which is not strictly separated.

Remark 12.4 It is reasonable to note that the pair $\{ A, B \}$ constructed in Theorem 12.1 is similar to the Sierpiṅski partition of the unit square $]0;1[^2$ (see, e.g., [163]).

Remark 12.5 Applying the well-known results of Cohen and Gödel (see [27] and [53]), we conclude that each of the following sentences:

-*"for an arbitrary probability space (E, S, μ) the μ-measure of the union of every family $(E_i)_{i\in I}$ of μ-measure zero subsets, such that card$(I) < c$, is equal to zero";*

-*"an arbitrary weakly separated family of probability measures is strictly separated, whenever its cardinality is not greater than* $2^{\aleph_0}$*"* ,

are independent of the theory ZFC.

Let us consider the question of transition from a weakly separated family of probability measures to the strictly separated one, when the family of probability measures is defined on the so called Radon metric space(About the notion of a Radon metric space, see, e.g., [86],[173]).

The next auxiliary proposition plays the key role in our further consideration.

Lemma 12.1 *Let* (E,ρ) *be a Radon metric space. Let* μ *be an arbitrary* σ*-finite Borel measure defined on* E*. Then there exists a closed separable subspace* $E(\mu)$ *of* E *such that*

$$\mu(E \setminus E(\mu)) = 0.$$

Remark 12.6 We remind the reader that a cardinal number α is real-valued measurable if there exists a continuous probability measure defined on the class of all subsets of some set of cardinality α. In connection with Lemma 12.1, we must also recall that an arbitrary complete metric space (E,ρ) whose topological weight is not a real-valued measurable cardinal, is a Radon metric space (cf. [88],p.48,Theorem 7).

The following important result is essentially due to Martin and Solovay (cf.[113]).

Lemma 12.2 *Let* (F,ρ) *be a separable complete metric space equipped with some probability Borel measure* μ*. If* $(E_i)_{i\in I}$ *is a family of* μ*-measure zero subsets of* F*, such that* $card(I) < c$*, then (in the system of axioms* (ZFC) & (MA)*) the outer measure* μ^* *of the set*

$$E = \bigcup_{i\in I} E_i$$

is equal to zero.

Proof. Let $\varepsilon > 0$. We must find an open set including E which has μ-measure $\leq \varepsilon$. Let P be the collection of all open sets of measure $< \varepsilon$; and for $p,q \in P$ let $p \preceq q$ mean $p \subseteq q$. We first show that the set P satisfies the $c.c.c.$ Let Q be a pairwise incompatible subset of P. Let

$$Q_n = \{p \in Q : m(p) \leq (1 - \frac{1}{2^n}\varepsilon)\}.$$

It is enough to show that Ω_n is countable.

Let $E' \subset F$ be a countable set everywhere dense in F.

For each $p \in Q_n$, choose $\overline{p} \subseteq p$ so that $\overline{p}$ is a finite union of open balls with center at E' and rational radius such that

$$\mu(p - \overline{p}) < \frac{\varepsilon}{2^n}.$$

Since all such finite unions constitute only a countable family, it suffices to show that if p and q are distinct members of Q_n, then $\overline{p} \neq \overline{q}$. Suppose $\overline{p} = \overline{q}$. Then

$$p \cup q \subseteq (p \setminus \overline{p}) \cup q,$$

so
$$\mu(p\cup q)<\frac{\varepsilon}{2^n}+(1-\frac{1}{2^n})\varepsilon=\varepsilon.$$

This implies that p and q are compatible, which contradicts $p,q\in\Omega$.

For $i\in I$, let $D_i=\{p\in P : E_i\subseteq p\}$. Using the relation $\mu(E_i)=0$, we easily conclude that D_i is dense in P. By Martin's axiom it follows that there is a subnet D of P which meets every D_i.

Let G be the union of the members of D. Then G is open. From $D_i\cap D\neq\emptyset$ we find that $E_i\subseteq G$, so $E\subseteq G$.

It remains to show that $\mu(G)>\varepsilon$ leads to a contradiction. By Lindelöf's theorem, G is a countable union of sets in D. It follows that there is a finite union G_1 of sets in D such that $\mu(G_1)>\varepsilon$. But since D is a subnet, some member of D includes G_1 and hence has measure $\geq\ \varepsilon$. This is a contradiction, and Lemma 12.2 is proved. $\square$

The following theorem is valid.

Theorem 12.2 *Let (F,ρ) be a Radon metric space. Let $(\mu_i)_{i\in I}$ be a weakly separated family of Borel probability measures with card$(I)\leq\mathbf{c}$, defined on (F,ρ).Then,in the system of axioms (ZFC) & (MA), the family $(\mu_i)_{i\in I}$ is strictly separated.*

Proof. Note that an arbitrary Borel probability measure μ defined on the space (F,ρ) has the property

$$(\forall J)(\forall (X_i)_{i\in J})((card(J)<2^{\aleph_0}\ \&\ (\forall i)(i\in J\rightarrow$$

$$\rightarrow\mu(X_i)=0)\rightarrow\mu^*(\bigcup_{i\in J}X_i)=0).$$

Indeed, by Lemma 12.1 applied to μ, there exists a separable closed support $F(\mu)$ in (F,ρ). Let us consider the set

$$\bigcup_{i\in J}X_i=[(\bigcup_{i\in J}X_i)\cap F(\mu)]\cup[(F\setminus F(\mu))\cap(\bigcup_{i\in J}X_i)].$$

Using Lemma 12.2, we conclude that the set $(\bigcup_{i\in J}X_i)\cap F(\mu)$ is a μ^*-measure zero subset of $F(\mu)$. Note that the outer measure of the set

$$(F\setminus F(\mu))\cap(\bigcup_{i\in J}X_i)$$

is equal to zero, because $\mu(F\setminus F(\mu))=0$.

Let $(\mu_i)_{i\in J}$ be a weakly separated family of Borel probability measures with card$(J)\leq\mathbf{c}$. Let us represent this family as an injective sequence $(\mu_\xi)_{\xi\prec\omega_\alpha}$, where the first ordinal number of cardinality J is denoted by ω_α. Since the family $(\mu_\xi)_{\xi\prec\omega_\alpha}$ is weakly separated, there exists a family $(X_\xi)_{\xi\prec\omega_\alpha}$ of Borel subsets of the space F, such that

$$(\forall\xi)(\forall\tau)(\xi\in[0;\omega_\alpha[\ \&\ \tau\in[0;\omega_\alpha[\rightarrow\mu_\xi(X_\tau)=\delta(\xi,\tau)),$$

where $\delta(.,.)$ denotes Kronecker's function on the Cartesian square $[0;\omega_\alpha[^2$ of the set $[0;\omega_\alpha[$. Let us define an ω_α-sequence of subsets $(B_\xi)_{\xi\prec\omega_\alpha}$ of the metric space F, such that:

1) $(\forall\xi)(\xi \prec \omega_\alpha \to\ B_\xi$ is a Borel subset in $F)$;
2) $(\forall\xi)(\xi \prec \omega_\alpha \to B_\xi \subseteq X_\xi)$;
3) $(\forall\tau_1)(\forall\tau_2)(\tau_1 \prec \omega_\alpha\ \&\ \tau_2 \prec \omega_\alpha\ \&\ \tau_1 \neq \tau_2 \to B_{\tau_1} \cap B_{\tau_2} = \emptyset)$;
4) $(\forall\tau)(\tau \prec \omega_\alpha \to \mu_\tau(B_\tau) = 1)$.

Take $B_0 = X_0$. Let, for $\xi \prec \omega_\alpha$, the partial sequence $(B_\tau)_{\tau\prec\xi}$ be already constructed. It is clear that

$$\mu_\xi^*(\bigcup_{\tau\prec\xi} B_\tau) = 0.$$

This means that there exists a Borel subset Y_ξ of the space F, such that

$$\bigcup_{\tau\prec\xi} B_\tau \subseteq Y_\xi,\ \mu_\xi(Y_\xi) = 0.$$

We put $B_\xi = X_\xi \setminus Y_\xi$. Now, it can easily be verified that the ω_α-sequence $(B_\xi)_{\xi\prec\omega_\alpha}$ of disjoint measurable subsets of the space F is constructed so that

$$(\forall\xi)(\xi \prec \omega_\alpha \to \mu_\xi(B_\xi) = 1). \qquad \square$$

Remark 12.7 Theorem 12.2 generalizes the main results obtained in [132] and [180]. Similar results are also discussed in [85],[88],[91],[127] and [131].

The next remark shows that all complete metric spaces can be assumed to be Radon(under some additional set-theoretical hypothesis).

Remark 12.8. The following conditions are equivalent:
a) an arbitrary complete metric space is a Radon space;
b) there does not exist a real-valued measurable cardinal.

Proof. $a) \to b)$. Assume the contrary and let J be a real-valued measurable cardinal. Let μ be a continuous probability measure defined on the class of all subsets of J.

Let us define a metric space (V,ρ) by:
1) $V = J$;
2) $(\forall x)(\forall y)(x \in V\ \&\ y \in V\ \to \rho(x,y) = 1$ if $x \neq y$, and $\rho(x,y) = 0$ if $x = y)$.

It is clear that (V,ρ) is a complete metric space whose topological weight is equal to J. The measure μ is not concentrated on a separable closed subset, because such a subset is at most countable and, hence, has μ-measure zero.

$b) \to a)$. Let (V,ρ) be an arbitrary complete metric space and W be its topological weight. By using the validity of the condition $b)$, we have that W is not a real-valued measurable cardinal. In view of Remark 12.6, we conclude that (V,ρ) is a Radon metric space. $\square$

Chapter 13

On Ostrogradsky's Formula in ℓ_2

In 1826, M.Ostrogradsky proved that if D is a non-empty region in $\mathbf{R}^3$ with boundary ∂D and $A=(A_x,A_y,A_z)$ is a continuously differentiable vector field, then the volume integral of the divergence div$A$ over D and the surface integral of $\mathbf{A}\cdot\mathbf{n}$ over the boundary ∂D are related by

$$\int_D \mathrm{div}\mathbf{A}dv=\int_{\partial D}\mathbf{A}\cdot\mathbf{n}ds, \tag{13.1}$$

where $\mathrm{div}\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_y}{\partial y}$, $\mathbf{n}(x,y,z)=(\cos(\alpha),\cos(\gamma),\cos(\beta))$ denote an external normed normal of the boundary ∂D at the point $(x,y,z)\in\partial D$ and $\mathbf{A}\cdot\mathbf{n}=A_x\cos(\alpha)+A_y\cos(\gamma)+A_z\cos(\beta)$, respectively. This mathematical statement describes the physical fact that, in the absence of the creation or destruction of matter, the density within a region of a space can change only by having its flow into or away from the region through its boundary.

In 1834, M.Ostrogradsky generalized his result in the case of n-dimensional Euclidean space $\mathbf{R}^n$. In particular, he has established the validity of the following formula

$$\int_D\sum_{k=1}^n\frac{\partial A_k}{\partial x_k}db_n=\int_{\partial D}\frac{\sum_{k=1}^n A_k\frac{\partial f}{\partial x_k}}{\sqrt{\sum_{k=1}^n(\frac{\partial f}{\partial x_k})^2}}ds, \tag{13.2}$$

where b_n denotes an n-dimensional classical Borel measure on $\mathbf{R}^n$, f is a continuously differentiable function on $\mathbf{R}^n$ which defines a boundary ∂D of the region $D\subset\mathbf{R}^n$ by the equation

$$f(x_1,\cdots,x_n)=0, \tag{13.3}$$

$A=(A_k)_{1\le k\le n}$ is a continuously differentiable vector field on $\mathbf{R}^n$.

Ostrogradsky's formula holds in all cases when the field $\mathbf{A}$ and its divergence div$\mathbf{A}$ do not approach infinity in D. It is also valid when the divergence approaches infinity in such a way that the integral in the left -hand of the formula (13.1) is convergent.

The physical meaning of the divergence of a field depends on the nature of the vector field $\mathbf{A}$. For instance, for the velocity field $\mathbf{v}$ of a gas flow div$\mathbf{v}$ is equal to the rate of relative expansion of an infinitesimal volume of gas and div$\rho\mathbf{v}$ is equal to the density of

mass sources. If the mass of the gass remains constant in the process of its flow we must have div$\rho\mathbf{v} = 0$ (in general case the mass can recieve an increment, positive or negative, resulting from a chemical or some other reaction in which the mass can change). At the same time we can have div$\mathbf{v} > 0$, div$\mathbf{v} < 0$ or div$\mathbf{v} = 0$ depending on whether the gas expands, contracts or does not change its density in the process of flow.

Our simple example demonstrates that Ostrogradsky's formula plays an important role in the theory of vector fields, when the vecor fields are finite-dimensional. Here naturally arises the following problem:

Whether Ostrogradsky's formula can be generalized in the case of an infinite-dimensional vector field defined on the Hilbert spase ℓ_2?

In this chapter we present one application of the method elaborated in Chapter 7 for the partial solution of this problem.

Here we preserve the main notations introduced in Chapter 7.

For arbitrary $\mathbb{J} \subset \mathbb{N}$, we put

$$\ell_2(\mathbb{J}) = \{ (x_k)_{k\in\mathbb{J}} \in \mathbb{R}^{\mathbb{J}} \ \& \sum_{k\in\mathbb{J}} x_k^2 < \infty \}, \tag{13.4}$$

$$T_{\mathbb{J}}((x_k)_{k\in\mathbb{J}}) = (\frac{x_k}{k+1})_{k\in\mathbb{J}}. \tag{13.5}$$

It is clear that, if $\mathrm{card}(\mathbb{J}) < \omega$, then

$$\ell_2(\mathbb{J}) = \mathbb{R}^{\mathbb{J}}. \tag{13.6}$$

One can easily demonstrate, that

$$B(\ell_2(\mathbb{J}_1 \cup \mathbb{J}_2)) = B(\ell_2(\mathbb{J}_1)) \times B(\ell_2(\mathbb{J}_2)) \tag{13.7}$$

for $\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{N}$ with $\mathbb{J}_1 \cap \mathbb{J}_2 = \emptyset$, where $B(\cdot)$ denotes a Borel σ-algebra of the corresponding space.

Lemma 13.1 *A functional $\mu_{\mathbb{J}}$, defined by*

$$(\forall X)(X \in B(\ell_2(\mathbb{J})) \rightarrow \mu_{\mathbb{J}}(X) = \nu_{\mathbb{J}}(T_{\mathbb{J}}^{-1}(X))) \tag{13.8}$$

for $\mathbb{J} \subseteq \mathbb{N}$, is a translation-invariant Borel measure in $\ell_2(\mathbb{J})$ which gets a numerical value one on the cube

$$\prod_{i\in\mathbb{J}} [0, \frac{1}{i+1}]. \tag{13.9}$$

Lemma 13.2 *Let $\mathbb{J}_1, \mathbb{J}_2 \subset \mathbb{N}$ such that $\mathbb{J}_1 \cap \mathbb{J}_2 = \emptyset$. Then the following condition*

$$(\forall X_1)(\forall X_2)(0 \leq \mu_{\mathbb{J}_1}(X_1) < \infty \ \& \ 0 \leq \mu_{\mathbb{J}_2}(X_2) < \infty \rightarrow$$

$$\rightarrow \mu_{\mathbb{J}_1}(X_1) \times \mu_{\mathbb{J}_2}(X_2) = \mu_{\mathbb{J}_1 \cup \mathbb{J}_2}(X_1 \times X_2)) \tag{13.10}$$

holds.

Proof. We have

$$
\begin{aligned}
&\mu_{\mathbb{J}_1\cup\mathbb{J}_2}(X_1\times X_2)=\nu_{\mathbb{J}_1\cup\mathbb{J}_2}(T^{-1}_{\mathbb{J}_1\cup\mathbb{J}_2}(X_1\times X_2))\\
&=\int\limits_{\prod_{k\in\mathbb{J}_1\cup\mathbb{J}_2}S_k} f_{T^{-1}_{\mathbb{J}_1\cup\mathbb{J}_2}(X_1\times X_2)}(g)d\overline{\lambda_{\mathbb{J}_1\cup\mathbb{J}_2}}((g_i)_{i\in\mathbb{J}_1\cup\mathbb{J}_2})\\
&=\int\limits_{\prod_{k\in\mathbb{J}_1\cup\mathbb{J}_2}S_k} f_{T^{-1}_{\mathbb{J}_1}(X_1)}((g_i)_{i\in\mathbb{J}_1})\times f_{T^{-1}_{\mathbb{J}_2}(X_2)}((g_i)_{i\in\mathbb{J}_2})d\overline{\lambda_{\mathbb{J}_1\cup\mathbb{J}_2}}((g_i)_{i\in\mathbb{J}_1\cup\mathbb{J}_2})\\
&=\int\limits_{\prod_{k\in\mathbb{J}_1}S_k} f_{T^{-1}_{\mathbb{J}_1}(X_1)}((g_i)_{i\in\mathbb{J}_1})d\overline{\lambda_{\mathbb{J}_1}}((g_i)_{i\in\mathbb{J}_1})\times\int\limits_{\prod_{k\in\mathbb{J}_2}S_k} f_{T^{-1}_{\mathbb{J}_2}(X_2)}((g_i)_{i\in\mathbb{J}_2})d\overline{\lambda_{\mathbb{J}_2}}((g_i)_{i\in\mathbb{J}_2})\\
&=\nu_{\mathbb{J}_1}(T^{-1}_{\mathbb{J}_1}(X_1))\times\nu_{\mathbb{J}_2}(T^{-1}_{\mathbb{J}_2}(X_2))=\mu_{\mathbb{J}_1}(X_1)\times\mu_{\mathbb{J}_2}(X_2).
\end{aligned}
\tag{13.11}
$$

This ends the proof of Lemma 13.2. □

Here we introduce some notions of the theory of vector fields in ℓ_2.

Let D_i be some Borel subset in $\ell_2(\mathbb{N}\setminus\{i\})$ with $0<\mu_{\mathbb{N}\setminus\{i\}}(D_i)<\infty$ and let

$$f_i : D_i \to \ell_2(\{i\})$$

be any real-valued function on D_i.

The set $\Gamma_{f_i}\subset\ell_2$, defined by

$$
\begin{aligned}
\Gamma_{f_i}=\{&(x_0,\cdots,x_{i-1},f_i((x_0,\cdots,x_{i-1},x_{i+1},\cdots)),x_{i+1},\cdots)\\
&:(x_0,\cdots,x_{i-1},x_{i+1},\cdots)\in D_i\}
\end{aligned}
\tag{13.12}
$$

is called a surface generated by the function f_i.

We say that f_i is differentiable at the point $(x_k)_{k\in\mathbb{N}\setminus\{i\}}$ if there exists $(C_k)_{k\in\mathbb{N}\setminus\{i\}}\in\ell_2(\mathbb{N}\setminus\{i\})$ and $\sigma:\ell_2(\mathbb{N}\setminus\{i\})\to\mathbb{R}$ with the property

$$\lim_{\sum_{k\in\mathbb{N},k\neq i}h_k^2\to 0}\sigma((h_0,\cdots,h_{i-1},h_{i+1},\cdots))=0, \tag{13.13}$$

such that an equality

$$
\begin{aligned}
&f_i((x_0+\Delta x_0,\cdots,x_{i-1}+\Delta x_{i-1},x_{i+1}+\Delta x_{i+1},\cdots))-f_i((x_0,\cdots,x_{i-1},x_{i+1},\cdots))\\
&=\sum_{k\in\mathbb{N},k\neq i}C_k\Delta x_k+\sigma(\Delta x_0,\cdots,\Delta x_{i-1},\Delta x_{i+1},\cdots)\sqrt{\sum_{k\in\mathbb{N},k\neq i}(\Delta x_k)^2}
\end{aligned}
\tag{13.14}
$$

holds for arbitrary

$$(\Delta x_k)_{k\in\mathbb{N},k\neq i}\in\ell_2(\mathbb{N}\setminus\{i\}) \tag{13.15}$$

If a function $g_k(x) = f_i(a_0,\cdots,a_{k-1},a_k+x,a_{k+1},\cdots))$ is differentiable at $0 \in \mathbb{R}$, then the value $\frac{dg_k}{dx}(0)$ is called a partial derivative of f_i with respect to the value x_k $(k \in \mathbb{N} \setminus \{ i\})$ at point $a = (a_j)_{j\in\mathbb{N}\setminus\{ i\}}$ and is denoted by $\frac{\partial f_i}{\partial x_k}(a)$.

Note that if $x_i = f_i((x_0,\cdots,x_{i-1},x_{i+1},\cdots))$ is differentiable in above-mentioned sense at the point $(a_j)_{j\in\mathbb{N}\setminus\{ i\}}$ then $C_k = \frac{\partial f_i}{\partial x_k}((a_j)_{j\in\mathbb{N}\setminus\{ i\}})$ for arbitrary $k \in \mathbb{N} \setminus \{ i\}$.

Note also that if $x_i = f_i((x_0,\cdots,x_{i-1},x_{i+1},\cdots))$ is differentiable at the point $(a_j)_{j\in\mathbb{N}\setminus\{ i\}}$, then a surface Γ_{f_i} has a tangential plane π_M at the point $M = (a_0,\cdots,a_{i-1}, f_i(a_0,\cdots,a_{i-1},a_{i+1},\cdots),a_{i+1},\cdots)$ which is defined by:

$$\pi_M = \{ (x_k)_{k\in\mathbb{N}} \big| (x_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N}) \;\&\; \sum_{k\in\mathbb{N}\setminus\{ i\}} \frac{\partial f_i}{\partial x_k}((a_k)_{k\in\mathbb{N}\setminus\{ i\}})(x_k - a_k) +$$

$$(x_i - f_i((a_0,\cdots,a_{i-1},a_{i+1},\cdots))) = 0\}. \tag{13.16}$$

We say that a Borel subset $D \subset \ell_2(\mathbb{N})$ is simple in the i-th direction if there exists a Borel subset $D_i \in \ell_2(\mathbb{N}\setminus \{ i\})$ with $0 < \mu_{\mathbb{N}\setminus\{ i\}}(D_i) < \infty$ and the differentiable functions $\Psi_1^{(i)}$ and $\Psi_2^{(i)}$ defined on D_i such that

1) $(\forall x)(x \in D_i \to \Psi_1^{(i)}(x) < \Psi_2^{(i)}(x))$;

2) $D = \{ (x_k)_{k\in\mathbb{N}} \big| (x_0,\cdots,x_{i-1},x_{i+1},\cdots) \in D_i \;\&$

$$\Psi_1^{(i)}((x_0,\cdots,x_{i-1},x_{i+1},\cdots)) < x_i < \Psi_2^{(i)}((x_0,\cdots,x_{i-1},x_{i+1},\cdots))\}. \tag{13.17}$$

A subset $D \subset \ell_2(\mathbb{N})$ is called simple if it is simple in the i-th direction for every $i \in \mathbb{N}$ and $S_i \cap S_j = \emptyset$ for $0 \le i < j < \infty$, where $S_k = \Gamma_{\Psi_1^{(k)}} \cup \Gamma_{\Psi_2^{(k)}}$ for $k \in \mathbb{N}$.

Let $M \in S_i$ and π_M be a tangential plane to the surface S_i at the point M. A normed vector n_M^+ is called an external normed normal of the surface S_i at the point M, if $\cos(\angle(n_M^+,e_i)) > 0$, $n_M^+ \perp \pi_M$ when $M \in \Gamma_{\Psi_2^{(i)}}$, or $\cos(\angle(n_M^+,e_i)) < 0$, $n_M^+ \perp \pi_M$ when $M \in \Gamma_{\Psi_1^{(i)}}$. A vector $-n_M^+$ is called an inner normed normal of the surface S_i at the point M and is denoted by n_M^-.

A simple Borel set D is called a cube-set if $n_{M_i}^+ \perp e_j$ for every different $i, j \in \mathbb{N}$ and $M_i \in S_i$.

Remark 13.1 Note that a set $\prod_{i\in\mathbb{N}}]x_i, x_i + \frac{1}{i+1}[$ is the cube-set in ℓ_2 for arbitrary $(x_i)_{i\in\mathbb{N}} \in \ell_2$.

Let $D \subset \ell_2(\mathbb{N})$ be a cube-set in above-mentioned sense.

A surface integral of the first order from the function $g : \ell_2(\mathbb{N}) \to R$ along i-th surface S_i of D is denoted by $\int_{S_i} g ds_i$ and is defined by

$$\int_{S_i} g ds_i = \int_{D_i} \frac{g(M_2)}{\cos(\angle(n_{M_2}{}^+, e_i))} d\mu_{\mathbb{N}\setminus\{i\}} - \int_{D_i} \frac{g(M_1)}{\cos(\angle(n_{M_1}{}^+, e_i))} d\mu_{\mathbb{N}\setminus\{i\}}, \tag{13.18}$$

where

$M_k = (x_0, \cdots, x_{i-1}, \Psi_k^{(i)}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots), x_{i+1}, \cdots)$ for $k = 1, 2$,

$n^+_{M_1} = (\frac{\partial \Psi_1^{(i)}}{\partial x_1}, \cdots, \frac{\partial \Psi_1^{(i)}}{\partial x_{i-1}}, -1, \frac{\partial \Psi_1^{(i)}}{\partial x_{i+1}}, \cdots)(M_1)$,

$n^+_{M_2} = (\frac{\partial \Psi_2^{(i)}}{\partial x_1}, \cdots, \frac{\partial \Psi_2^{(i)}}{\partial x_{i-1}}, 1, \frac{\partial \Psi_2^{(i)}}{\partial x_{i+1}}, \cdots)(M_2)$.

If the series $\sum_{i=1}^{\infty} \int_{S_i} g ds_i$ is convergent then its sum is called a surface integral of the first order from the function $g : \ell_2(\mathbb{N}) \to \mathbf{R}$ along the surface ∂D and is denoted by $\int_{\star \partial D} g ds$, i.e.,

$$\int_{\star \partial D} g ds = \sum_{i=1}^{\infty} \int_{S_i} g ds_i. \tag{13.19}$$

A surface integral of the second order from the function $g : \ell_2(\mathbb{N}) \to R$ along the surface ∂D (in the i-th direction) is denoted by $\int_{\partial D} g d\mu_{\mathbb{N}\setminus\{i\}}$ and is defined by

$$\int_{\partial D} g d\mu_{\mathbb{N}\setminus\{i\}} = \int_{D_i} g((x_0, \cdots, x_{i-1}, \Psi_2^{(i)}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots), x_{i+1}, \cdots)) d\mu_{\mathbb{N}\setminus\{i\}} -$$

$$\int_{D_i} g((x_0, \cdots, x_{i-1}, \Psi_1^{(i)}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots), x_{i+1}, \cdots)) d\mu_{\mathbb{N}\setminus\{i\}}. \tag{13.20}$$

We need the following auxiliary proposition, which can be considered as an analog of Fubini theorem in ℓ_2.

Lemma 13.3 *Let a Borel set D be simple in i-th direction and g be a $\mu_{\mathbb{N}}$-integrable function defined on D. Then the following formula is valid*

$$\int_D g d\mu_{\mathbb{N}} = \int_{D_i} \Big(\int_{\Psi_1^{(i)}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots)}^{\Psi_2^{(i)}(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots)} g d\mu_{\{i\}}\Big) d\mu_{\mathbb{N}\setminus\{i\}}. \tag{13.21}$$

Proof. We put $\nu_i(\cdot) = \mu_{\mathbb{N}\setminus\{i\}}(D_i) \times \mu_{\mathbb{N}\setminus\{i\}}(\cdot \,|D_i)$, where $\mu_{\mathbb{N}\setminus\{i\}}(\cdot \,|D_i)$ denotes a conditional measure. Let us consider a product measure $\mu_{\{i\}} \times \nu_i$.

Let show that

$$(\forall X)(X \in B(D_i \times \ell_2(\{i\})) \to \mu_{\mathbb{N}}(X) = (\mu_{\{i\}} \times \nu_i)(X)). \tag{13.22}$$

It is clear that the sets of the form $X_1 \times X_2$, where $X_1 \in B(\ell_2(\{i\}))$ and $X_2 \in B(D_i)$, generate a σ-algebra of subsets of $D_i \times \ell_{\{i\}}$. Actually,

$$(\mu_{\{i\}} \times \nu_i)(X_1 \times X_2) = \mu_{\{i\}}(X_1) \times \nu_i(X_2) = \mu_{\{i\}}(X_1) \times \mu_{\mathbb{N}\setminus\{i\}}(X_2). \tag{13.23}$$

Using Lemma 13.2, we have

$$\mu_{\{i\}}(X_1) \times \mu_{\mathbb{N}\setminus\{i\}}(X_2) = \mu_{\mathbb{N}}(X_1 \times X_2). \tag{13.24}$$

Now, using Caratheodory's well-known result we conclude that

$$(\forall X)(X \in B(D_i \times \ell_2(\{i\})) \to \mu_{\mathbb{N}}(X) = (\mu_{\{i\}} \times \nu_i)(X)). \tag{13.25}$$

Consequently,

$$(\forall X)(X \in B(D) \to \mu_{\mathbb{N}}(X) = (\mu_{\{i\}} \times \nu_i)(X)). \tag{13.26}$$

Using Fubini theorem, we establish that

$$\int_{D_i} \left(\int_{\Psi_1^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)}^{\Psi_2^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)} g d\mu_{\{i\}} \right) d\mu_{\mathbb{N}\setminus\{i\}} =$$

$$\int_{D_i} \left(\int_{\Psi_1^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)}^{\Psi_2^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)} g d\mu_{\{i\}} \right) d\nu_i = \int_D g d(\mu_{\{i\}} \times \nu_i) = \int_D g d\mu_{\mathbb{N}}. \tag{13.27}$$

This ends the proof of Lemma 13.3. □

The main result is formulated as follows.

Theorem 13.1 *Let D be a cube-set in $\ell_2(\mathbb{N})$ and let $A = (A_i)_{i\in\mathbb{N}}$ be a continuously differentiable vector field in ℓ_2 such that*

$$\int_D \sum_{i=1}^{\infty} \frac{\partial A_i}{\partial x_i} d\mu_{\mathbb{N}} = \sum_{i=1}^{\infty} \int_D \frac{\partial A_i}{\partial x_i} d\mu_{\mathbb{N}}. \tag{13.28}$$

Then the following formula is valid

$$\int_D divA d\mu_{\mathbb{N}} = \int_{\star\partial D} < T_{\mathbb{N}}^{-1}(A), \mathbf{n}_{\mathbf{M}}^{+} >_{\ell_2} ds, \tag{13.29}$$

where $< \cdot, \cdot >_{\ell_2}$ denotes an usual scalar product in ℓ_2.

Proof. Using Lemma 13.3, we have

$$\int_D \frac{\partial A_i}{\partial x_i} d\mu_{\mathbb{N}} = \int_{D_i} \Big(\int_{\Psi_1^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)}^{\Psi_2^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots)} \frac{\partial A_i}{\partial x_i} d\mu_{\{i\}} \Big) d\mu_{\mathbb{N}\setminus\{i\}} =$$

$$= \int_{D_i} (i+1)(A_i(x_0,\cdots,x_{i-1},\Psi_2^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots),x_{i+1},\cdots) -$$

$$-A_i(x_0,\cdots,x_{i-1},\Psi_1^{(i)}(x_0,\cdots,x_{i-1},x_{i+1},\cdots),x_{i+1},\cdots)) d\mu_{\mathbb{N}\setminus\{i\}} =$$

$$= (i+1) \times \int_{D_i} \Big(\frac{A_i(M_1)\cos(\angle(n_{M_1}{}^{+}, e_i))}{\cos(\angle(n_{M_1}{}^{+}, e_i))} - \frac{A_i(M_2)\cos(\angle(n_{M_2}{}^{+}, e_i))}{\cos(\angle(n_{M_2}{}^{+}, e_i))} \Big) d\mu_{\mathbb{N}\setminus\{i\}}$$

$$= \int_{S_i} (i+1)A_i \cos(\angle(n_M{}^+, e_i))ds_i$$
$$= \sum_{k\geq 1} \int_{S_k} (i+1)A_i \cos(\angle(n_M{}^+, e_i))ds_k$$
$$= \int_{\star\partial D} (i+1)A_i \cos(\angle(n_M{}^+, e_i))ds. \tag{13.30}$$

Following (13.28) and (13.30), we have

$$\int_D \mathbf{div} A d\mu_{\mathbb{N}} = \int_D \sum_{i=1}^{\infty} \frac{\partial A_i}{\partial x_i} d\mu_{\mathbb{N}} = \sum_{i=1}^{\infty} \int_D \frac{\partial A_i}{\partial x_i} d\mu_{\mathbb{N}}$$
$$= \sum_{i=1}^{\infty} \int_{\star\partial D} (i+1)A_i \cos(\angle(n_M{}^+, e_i))ds$$
$$= \int_{\star\partial D} \sum_{i=1}^{\infty} (i+1)A_i \cos(\angle(\mathbf{n_M}{}^+, e_i))ds$$
$$= \int_{\star\partial D} < T_{\mathbb{N}}^{-1}(A), \mathbf{n_M}{}^+) >_{\ell_2} ds. \tag{13.31}$$

This ends the proof of Theorem 13.1. □

Remark 13.2 Under our notations, the formula (13.2) can be rewritten as follows

$$\int_D \mathrm{div} \mathbf{A} d\mu_{\{1,\cdots,n\}} = \int_{\star\partial D} < T_{\{1,\cdots,n\}}^{-1}(A), \mathbf{n}_{\mathbf{M}}^{+} >_{\ell_2(\{1,\cdots,n\})} ds, \tag{13.32}$$

where $< \cdot, \cdot >_{\ell_2(\{1,\cdots,n\})}$ denotes an usual scalar product in $\ell_2(\{1,\cdots,n\}) = \mathbb{R}^n$. By this reason the formula (13.29) can be considered as a generalization of Ostrogradsky's formula (13.2) for cube-sets in ℓ_2.

In context of Theorem 13.1, the following simple example is of interest.

Example 13.1 Let $\mathbf{A}$ be a field in ℓ_2 defined by

$$\mathbf{A}((x_i)_{i\in N}) = (\frac{\lambda^i x_i}{i!})_{i\in N}, \tag{13.33}$$

and let $D = \Delta_b$, where

$$\Delta_b = \prod_{i\in N} [b_i - \frac{1}{2(i+1)}, b_i + \frac{1}{2(i+1)}] \tag{13.34}$$

for $b = (b_i)_{i\in N} \in \ell_2$.

Then, following Theorem 13.1, we have

$$\int_{\star\partial\ \Delta_b} < T_{\mathbb{N}}^{-1}(A), \mathbf{n_M}{}^+) >_{\ell_2} ds = \int_{\Delta_b} \sum_{i=1}^{\infty} \frac{\partial \mathbf{A}_i}{\partial x_i} d\mu_N =$$
$$\int_{\Delta_b} \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} d\mu_N = (e^{\lambda} - 1) \int_{\Delta_b} d\mu_N = e^{\lambda} - 1. \tag{13.35}$$

Chapter 14

On Generalized Fourier Series

In this chapter we present one application of a partial analog of a Lebesgue measure in Banach spaces with basis. In particular, in the Solovay model we introduce generalized Fourier series in infinite-dimensional separable Banach space with basis and investigate some of its properties.

Here we preserve the notations given in Chapter 7.

Lemma 14.1 *Let* $C_\beta = h_\beta + [0,1[^{\mathbb{N}\setminus\{k\}}$ *for* $\beta \in [0,x_k[$ $(x_k > 0)$ *and* $h_\beta \in \mathbb{R}^{\mathbb{N}\setminus\{k\}}$*. Then*

$$f_{\sum_{\beta\in[0,x_k[}\{\beta\}\times C_\beta}(g_k,(g_i)_{i\in\mathbb{N}\setminus\{k\}}) = \#(f_k^{-1}(g_k)\cap[0,x_k[) \tag{14.1}$$

for arbitrary $(g_k,(g_i)_{i\in\mathbb{N}\setminus\{k\}}) \in S_k \times \prod_{i\in\mathbb{N}\setminus\{k\}} S_i$*.*

Proof. Note that if $\beta \in f_k^{-1}(g_k)\cap[0,x_k[$, then

$$\#((f_k^{-1}(g_k)\times(\prod_{i\in N\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}}))\cap(\{\beta\}\times C_\beta)) =$$

$$\#((\prod_{i\in N\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}})\cap C_\beta) = 1. \tag{14.2}$$

If $\beta \in [0,x_k[\setminus f_k^{-1}(g_k)$, then

$$\#\Big((f_k^{-1}(g_k)\times(\prod_{i\in N\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}}))\cap(\{\beta\}\times C_\beta)\Big) = \#(\emptyset) = 0. \tag{14.3}$$

Hence, we have

$$f_{\sum_{\beta\in[0,x_k[}\{\beta\}\times C_\beta}(g_k,(g_i)_{i\in\mathbb{N}\setminus\{k\}}) =$$

$$\#((f_k^{-1}(g_k)\times(\prod_{i\in\mathbb{N}\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}}))\cap(\sum_{\beta\in[0,x_k[}\{\beta\}\times C_\beta)) =$$

$$\#((f_k^{-1}(g_k)\times(\prod_{i\in\mathbb{N}\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}}))\cap(\sum_{\beta\in f_k^{-1}(g_k)\cap[0,x_k[}\{\beta\}\times C_\beta))+$$

$$\#((f_k^{-1}(g_k)\times(\prod_{i\in\mathbb{N}\setminus\{k\}} f_i^{-1})((g_i)_{i\in\mathbb{N}\setminus\{k\}}))\cap(\sum_{\beta\in[0,x_k[\setminus f_k^{-1}(g_k)}\{\beta\}\times C_\beta))=$$

$$\#(f_k^{-1}(g_k)\cap[0,x_k[). \tag{14.4}$$

This ends the proof of Lemma 14.1. □

Lemma 14.2 *Let B be an infinite-dimensional separable Banach space with absolutely convergent basis $\Gamma=(e_k)_{k\in\mathbb{N}}$. We set*

$$A(h)=(x_k)_{k\in\mathbb{N}} \tag{14.5}$$

for $h=\sum_{k\in\mathbb{N}}x_ke_k\in B$. Then in Solovay's model, a functional μ_Γ, defined by

$$\mu_\Gamma(X)=\mu_{\mathbb{N}}(A(X)) \tag{14.6}$$

for $X\subseteq B$, is a translation-invariant diffused measure which gets a value one on the set $\Delta=\{\sum_{k\in\mathbb{N}}\alpha_ke_k:(\alpha_k)_{k\in\mathbb{N}}\in[0,1[^{\mathbb{N}}\}$.

Proof. The functional μ_Γ is a measure, because

$$\mu_\Gamma(\sum_{k\in\mathbb{N}}X_k)=\mu_{\mathbb{N}}(A(\sum_{k\in\mathbb{N}}X_k))=\sum_{k\in\mathbb{N}}\mu_{\mathbb{N}}(A(X_k))=\sum_{k\in\mathbb{N}}\mu_\Gamma(X_k). \tag{14.7}$$

The functional μ_Γ is translation-invariant. Indeed, for $h\in B$ we have

$$\mu_\Gamma(X+h)=\mu_{\mathbb{N}}(A(X+h))=\mu_{\mathbb{N}}(A(X)+A(h))=\mu_{\mathbb{N}}(A(X))=\mu_\Gamma(X). \tag{14.8}$$

Since $\mu_{\mathbb{N}}$ is the diffused measure, for $h=\sum_{k\in\mathbb{N}}x_ke_k\in B$ we have

$$\mu_\Gamma(\{h\})=\mu_{\mathbb{N}}(\{A(h)\})=\mu_{\mathbb{N}}(\{(x_k)_{k\in\mathbb{N}}\})=0. \tag{14.9}$$

The last relation means that μ_Γ is also a diffused measure.
Clearly,

$$\mu_\Gamma(\Delta)=\mu_N(A(\Delta))=\mu_{\mathbb{N}}([0,1[^{\mathbb{N}})=1. \tag{14.10}$$

This ends the proof of Lemma 14.2. □

Let $\mathbb{J}\subseteq\mathbb{N}$. We set $B_{\mathbb{J}}=\{b=\sum_{k\in\mathbb{N}}\alpha_ke_k: b\in B\ \&\ \alpha_k\geq 0 \text{ if } k\in\mathbb{J} \text{ and } \alpha_k<0 \text{ if } k\notin\mathbb{J}\}$. It is clear that $B=\sum_{\mathbb{J}\subseteq\mathbb{N}}B_{\mathbb{J}}$.

Let $(b_k)_{k\in\mathbb{N}}$ be a family of elements in B, such that

$$\sum_{k\in\mathbb{N}}||b_k||<\infty. \tag{14.11}$$

We set

$$P(b_1,\cdots)=\{\sum_{k\in\mathbb{N}}\alpha_kb_k:(\alpha_k)_{k\in\mathbb{N}}\in[0,1[^{\mathbb{N}}\}. \tag{14.12}$$

Clearly, $P(b_1,\cdots)$ can be considered as an infinite-dimensional parallelepiped in B generated by the family of vectors $(b_k)_{k\in\mathbb{N}}$.

Let $b \in B$. A formal series

$$\sum_{k\in N}(-1)^{\mathrm{Ind}_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))e_k \tag{14.13}$$

and a sequence of real numbers

$$((-1)^{\mathrm{Ind}_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots)))_{k\in\mathbb{N}}\,, \tag{14.14}$$

where $\mathbb{J}$ is a unique subset of $\mathbb{N}$ for which $b \in B_{\mathbb{J}}$ and $\mathrm{Ind}_{\mathbb{J}}(\cdot)$ denotes an indicator of the set $\mathbb{J}$ defined on $\mathbb{N}$, are called a generalized Fourier series and a family of generalized Fourier coefficients of the element b in the basis Γ, respectively.

Theorem 14.1 *Let B be an infinite-dimensional Banach space with absolutely convergent basis $\Gamma=(e_k)_{k\in\mathbb{N}}$ and let $b\in B$. Then, in Solovay's model, the generalized Fourier series of the element $b \in B$ in the basis Γ coincides with its representation in the same basis.*

Proof. Let $b=\sum_{k\in\mathbb{N}}x_ke_k$. By the definition of the infinite-dimensional parallelepiped $P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots)$ we have

$$P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots)=\cup_{(\alpha_i)_{i\in N}\in[0,1[^N}\{\sum_{i\in N\setminus\{k\}}\alpha_ie_i+\alpha_k\sum_{i\in N}x_ie_i\}=$$
$$\cup_{\alpha_k\in[0,1[}\cup_{(\alpha_i)_{i\in N\setminus\{k\}}\in[0,1[^{N\setminus\{k\}}}\{\sum_{i\in N\setminus\{k\}}\alpha_ie_i+\alpha_k\sum_{i\in N\setminus\{k\}}x_ie_i+\alpha_kx_ke_k\}=$$
$$\cup_{\alpha_k\in[0,1[}((\cup_{(\alpha_i)_{i\in N\setminus\{k\}}\in[0,1[^{N\setminus\{k\}}}\{\sum_{i\in N\setminus\{k\}}\alpha_ie_i\}+\alpha_k\sum_{i\in N\setminus\{k\}}x_ie_i)+\alpha_kx_ke_k). \tag{14.15}$$

Hence, we have

$$A(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))=\cup_{\alpha_k\in[0,1[}\{\alpha_kx_k\}\times([0,1[^{N\setminus\{k\}}+\alpha_k(x_i)_{i\in N\setminus\{k\}}). \tag{14.16}$$

If $x_k=0$, then we have

$$\mu_\Gamma(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))=\mu_{\mathbb{N}}(A((P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))=$$
$$\mu_{\mathbb{N}}(\cup_{\alpha_k\in[0,1[}\{0\}\times([0,1[^{\mathbb{N}\setminus\{k\}}+\alpha_k(x_i)_{i\in\mathbb{N}\setminus\{k\}}))\le\mu_{\mathbb{N}}(\{0\}\times\mathbb{R}^{\mathbb{N}\setminus\{k\}})=0=x_k. \tag{14.17}$$

Now assume that $x_k>0$. By the definition of the measure μ_Γ, we have

$$\mu_\Gamma(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))=\mu_{\mathbb{N}}(A((P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))=$$
$$\mu_{\mathbb{N}}(\sum_{\alpha_k\in[0,1[}\{x_k\alpha_k\}\times([0,1[^{N\setminus\{k\}}+\alpha_k(x_i)_{i\in\mathbb{N}\setminus\{k\}})). \tag{14.18}$$

We set

$$D_{\alpha_k} = [0,1[^{\mathbb{N}\setminus\{k\}} + \alpha_k (x_i)_{i\in\mathbb{N}\setminus\{k\}}. \tag{14.19}$$

We have

$$\mu_{\mathbb{N}}(A(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots)) = \int_{\prod_{i\in\mathbb{N}} S_i} f_{\sum_{\alpha_k\in[0,1[}(\{\alpha_k x_k\}\times D_{\alpha_k})}(g)d\overline{\lambda}_{\mathbb{N}}(g). \tag{14.20}$$

Using Fubini theorem and Lemma 14.1, we have

$$\int_{\prod_{i\in\mathbb{N}} S_i} f_{\sum_{\alpha_k\in[0,1[}(\{\alpha_k x_k\}\times D_{\alpha_k})}(g)d\overline{\lambda}_{\mathbb{N}}(g) =$$

$$\int_{S_k}[\int_{\prod_{i\in\mathbb{N}\setminus\{k\}} S_i} f_{\sum_{\beta_k\in[0,x_k]}\{\beta_k\}\times D_{\frac{\beta_k}{x_k}}}(g_k,(g_i)_{i\in\mathbb{N}\setminus\{k\}})d\overline{\lambda}_{N\setminus\{k\}}((g_i)_{i\in\mathbb{N}\setminus\{k\}})]d\overline{\lambda}_{\{k\}}(g_k) =$$

$$\int_{S_k}[\int_{\prod_{i\in\mathbb{N}\setminus\{k\}} S_i} f_{\sum_{\beta_k\in[0,x_k]}\{\beta_k\}\times C_{\beta_k}}(g_k,(g_i)_{i\in\mathbb{N}\setminus\{k\}})d\overline{\lambda}_{\mathbb{N}\setminus\{k\}}((g_i)_{i\in\mathbb{N}\setminus\{k\}})]d\overline{\lambda}_{\{k\}}(g_k) =$$

$$\int_{S_k}[\int_{\prod_{i\in\mathbb{N}\setminus\{k\}} S_i} \#(f_k^{-1}(g_k)\cap[0,x_k[)d\overline{\lambda}_{\mathbb{N}\setminus\{k\}}((g_i)_{i\in\mathbb{N}\setminus\{k\}})]d\overline{\lambda}_{\{k\}}(g_k) =$$

$$\int_{S_k} \#(f_k^{-1}(g_k)\cap[0,x_k[)[\int_{\prod_{i\in\mathbb{N}\setminus\{k\}} S_i} d\overline{\lambda}_{\mathbb{N}\setminus\{k\}}((g_i)_{i\in\mathbb{N}\setminus\{k\}})]d\overline{\lambda}_{\{k\}}(g_k) =$$

$$\int_{S_k} \#(f_k^{-1}(g_k)\cap[0,x_k[)d\overline{\lambda}_{\{k\}}(g_k) = |x_k|. \tag{14.21}$$

The validity of the formula (14.21) can be obtained analogously for $x_k < 0$.

This ends the proof of Theorem. □

Remark 14.1 *Note that if there exists a basis $\Gamma^* = (e_k^*)_{k\in\mathbb{N}}$ in B, then always exists an absolutely convergent basis $\Gamma = (e_k)_{k\in\mathbb{N}}$. In this direction we can set $\Gamma = (e_k)_{k\in\mathbb{N}} = (\frac{1}{2^k||e_k^*||}e_k^*)_{k\in\mathbb{N}}$. If $(x_k)_{k\in\mathbb{N}}$ are coefficients of the representation of the element $b\in B$ in the basis Γ, then $(y_k)_{k\in\mathbb{N}}$, defined by*

$$y_k = 2^k||e_k^*||x_k \quad (k\in\mathbb{N}), \tag{14.22}$$

are coefficients of the representation of the element $b\in B$ in the basis Γ^*.

Corollary 14.1 *Let B be an infinite-dimensional separable Banach space with a basis $\Gamma = (e_k)_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}}||e_k|| < \infty$. Then in Solovay's model for $b\in B$ we have*

$$b = \sum_{k\in\mathbb{N}}(-1)^{Ind_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1,\cdots,e_{k-1},b,e_{k+1},\cdots))e_k, \tag{14.23}$$

where $\mathbb{J}$ is a unique subset of $\mathbb{N}$ for which $b\in B_{\mathbb{J}}$.

Let L_k be a vector space in B, generated by the vector e_k for $k\in\mathbb{N}$ and let $L_k^\perp$ be a cospace of L_k. Let $< e_k,\cdot >$ be a linear functional which gets a numerical value one on e_k

and which is identically equal to zero on $L_k^{\perp}$, respectively. Then, in Solovay's model, we have

$$(\forall b)(b \in B \rightarrow < e_k, b >= (-1)^{Ind_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1, \cdots, e_{k-1}, b, e_{k+1}, \cdots))). \tag{14.24}$$

Corollary 14.2 *Let $(H, < \cdot >_H)$ be an infinite-dimensional separable Hilbert space with an orthogonal basis $\Gamma = (e_k)_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} ||e_k|| < \infty$. Then, in Solovay's model, for $h \in H$ we have*

$$< e_k, h >_H = ||e_k||^2(-1)^{Ind_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1, \cdots, e_{k-1}, h, e_{k+1}, \cdots)), \tag{14.25}$$

where $\mathbb{J}$ is a unique subset of $\mathbb{N}$ for which $b \in H_{\mathbb{J}}$.

Corollary 14.3 *Let R^n be a finite-dimensional Euclidian vector space with a basis $\Gamma = (e_k)_{1\le k\le n}$. Then for $b \in R^n$ we have*

$$b = \sum_{k=1}^{n}(-1)^{Ind_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1, \cdots, e_{k-1}, b, e_{k+1}, \cdots, e_n))e_k, \tag{14.26}$$

where $\mathbb{J}$ is a unique subset of $\{1, \cdots, n\}$ for which $b \in R^n_{\mathbb{J}}$, μ_Γ is a Lebesgue measure which obtains a numerical value of one on the parallelepiped $P(e_1, \cdots, e_n)$.

Remark 14.4 Note that (14.26) is equivalent to the well-known Crammer's formulas

$$x_k = \frac{\det(e_1, \cdots, e_{k-1}, b, e_{k+1}, \cdots, e_n)}{\det(e_1, \cdots, e_n)} \ (1 \le k \le n), \tag{14.27}$$

where $\det(a_1, \cdots, a_n)$ denotes the determinant of the matrix defined by the family of vectors $\{a_k : 1 \le k \le n\} \subset \mathbf{R}^n$.

Indeed, one can easily check the validity of the following equality

$$|\det(b_1, \cdots, b_n)| = \mu_n(P(e_1, \cdots, e_k))\mu_\Gamma(P(b_1, \cdots, b_k)), \tag{14.28}$$

where μ_n denotes a standard Lebesgue measure on $\mathbb{R}^n$.

Using (14.28) and the simple fact that there exists a unique subset $\mathbb{J} \subset \{1, \cdots, n\}$ such that $b \in R^n_{\mathbb{J}}$, (14.27) can be rewritten in the equivalent form

$$x_k = (-1)^{\mathrm{Ind}_{\mathbb{J}}(k)+1}\mu_\Gamma(P(e_1, \cdots, e_{k-1}, b, e_{k+1}, \cdots, e_n)) \tag{14.29}$$

for $k \in \{1, \cdots, n\}$, where $\mathrm{Ind}_{\mathbb{J}}(\cdot)$ denotes an indicator of the set $\mathbb{J}$ defined on $\{1, \cdots, n\}$.

Remark 14.5 By using the technique elaborated in Appendix 15.3 and in papers [2],[3], one can obtain analogous results in the system of axioms ZFC.

Chapter 15

Appendix

15.1. On one problem of J.R.Christensen

We begin our discussion with some notions of "small" sets in abelian Polish groups introduced by J.R.Christensen in [28].

Let $(G,+,\rho)$ be an abelian Polish group and let L be the class of all Borel probability measures on G. Let $\overline{\mu}$ denote a completion of $\mu \in L$.

Definition 15.1.1 *A class $U(G)$ of subsets of G, defined by*

$$U(G) = \bigcap_{\mu \in L} dom(\overline{\mu}),$$

where $dom(\overline{\mu})$ denotes a domain of $\overline{\mu}$, is called a class of all universally measurable subsets of the abelian Polish group $(G,+,\rho)$.

Definition 15.1.2 *Following [28, p.30], a universally measurable set S is called a Haar zero set if there exists a probability measure μ on $U(G)$ such that $\mu(S+h)=0$ for $h \in G$. The measure μ is called "testing" measure for S.*

In 1973 J.R. Christensen posed the following

Problem 15.1.1 ([28],Problem II,p.38). *Let $(A_i)_{i \in I}$ be a family of universally measurable pairwise disjoint subsets of the abelian Polish group $(G,+,\rho)$ which are not Haar zero sets. Is then I at most countable?*

Since in locally compact abelian Polish group G the notion of "Haar zero sets" coincides with the notion of "Haar measure zero sets"(cf.[28]), and the Haar measure on G is σ-finite, we easily conclude that the answer on above-mentioned problem in such groups is positive.

Now let N denote the set of all natural numbers.

Let R^∞ be a group of all real-valued sequences defined on N with the usual addition operation "+ ". Let a metric ρ on R^∞ be defined by

$$(\forall (x_i)_{i\in N}, (y_i)_{i \in N} \in R^\infty \rightarrow \rho((x_i)_{i\in N}, (y_i)_{i\in N}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)}).$$

Note that then $(R^\infty,+,\rho)$ is an infinite-dimensional abelian Polish group.

The purpose of the present section is to give a negative answer on the above-mentioned problem in $(R^\infty,+,\rho)$.

In the sequel we need some auxiliary propositions.

Lemma 15.1.1 *Let μ be an arbitrary probability measure on $U(R^\infty)$. Then there exists a compact subset $K \subset R^\infty$ such that $\mu(K) > 0$.*

This lemma immediately follows from the inner regularity of Borel probability measures in Polish spaces(cf. [86]) and the simple fact that $K \in U(R^\infty)$.

Lemma 15.1.2 *Let K be any compact set in $(R^\infty, +, \rho)$. Then there exists a family of closed intervals $([a_i, b_i])_{i \in N}$ such that*

$$K \subset \prod_{i \in N} [a_i; b_i].$$

Proof. For arbitrary $i \in N$ we set $U_{(i,k)} =]-k; k[\times R^{N \setminus \{i\}}$. It is clear that $(U_{(i,k)})_{k \in N}$ is the family of open sets in $(R^\infty, +, \rho)$ such that

$$R^\infty = \cup_{k \in N} U_{(i,k)}.$$

Since $(U_{(i,k)})_{k \in N}$ covers K, for $i \in \mathbb{N}$ there exist $k_i \in \mathbb{N}$ such that $K \subset]-k_i; k_i[\times R^{N \setminus \{i\}} (i \in \mathbb{N})$. If we set $[a_i; b_i] = [-k_i - 1; k_i + 1]$, then we obtain

$$K \subset \cap_{i \in N}]-k_i; k_i[\times R^{N \setminus \{i\}} \subset \cap_{i \in N} [a_i; b_i] \times R^{N \setminus \{i\}} = \prod_{i \in N} [a_i; b_i].$$

This ends the proof of Lemma 15.1.2. □

Lemma 15.1.3 *Let J be an arbitrary subset of N. We set*

$$A_J = \{(x_i)_{i \in N} : x_i \geq 0 \textit{ for } i \in J \ \& \ x_i < 0 \textit{ for } i \in N \setminus J\}.$$

Then A_J is an universally measurable (moreover,Borel) subset in $(R^\infty, +, \rho)$ which is not a Haar null set.

Proof. We set

$$(\forall i)(i \in N \rightarrow B_i = \left\{ \begin{array}{ll} [0, \infty[, & \text{if } i \in J, \\ ]-\infty, 0[, & \text{if } i \in N \setminus J \end{array} \right),$$

$$(\forall i)(i \in N \rightarrow C_i = R^{N \setminus \{i\}} \times B_i).$$

It is obvious that C_i is a Borel subset in $(R^\infty, +, \rho)$ for $i \in \mathbb{N}$ as well their intersection $\cap_{i \in N} C_i$, which exactly coincides with A_J. Since an arbitrary Borel subset is universally measurable, we deduce that A_J is universally measurable.

Now let assume the contrary and let μ be a "testing" measure for A_J. By using Lemma 15.1.1, we conclude that there exists a compact set K in $(R^\infty, +, \rho)$ such that $\mu(K) > 0$. By using Lemma 15.1.2, there exists a family of closed intervals $([a_i, b_i])_{i \in N}$ such that

$$K \subset \prod_{i \in N} [a_i; b_i].$$

We set

$$(\forall i)(i \in N \rightarrow h_i^{(0)} = \left\{ \begin{array}{ll} -\max\{|a_i| + 1, |b_i| + 1\}, & \text{if } i \in J \\ \max\{|a_i| + 1, |b_i| + 1\}, & \text{if } i \in N \setminus J \end{array} \right).$$

Then, for $h_0 = (h_i^{(0)})_{i \in N}$ we get

$$\mu(A_J + h_0) \geq \mu((A_J + h_0) \cap K) = \mu(K) > 0,$$

which contradicts to the assumption that μ is a "testing" measure for A_J.

This ends the proof of Lemma 15.1.3. $\square$

Our main result is formulated as follows.

Theorem 15.1.1 $\Phi = \{A_J : J \subseteq N\}$ *is the continual family of universally measurable (moreover, Borel) pairwise disjoint subsets in the abelian Polish group* $(R^\infty, +, \rho)$ *every element of which is not Haar zero.*

Proof. By using Lemma 15.1.3, we must show only that Φ is a family of pairwise disjoint subsets and the cardinality of Φ is equal to c, where c denotes the cardinality of the continuum.

Assume the contrary and let A_{J_1} and A_{J_2} be two different elements of Φ such that $A_{J_1} \cap A_{J_2} \neq \emptyset$. Without loss of generality we can assume that $J_1 \setminus J_2 \neq \emptyset$. Let $h = (h_i)_{i \in N} \in A_{J_1} \cap A_{J_1}$ and $i_0 \in J_1 \setminus J_2$. Then for h_{i_0} we get , $h_{i_0} \geq 0$ (because $h \in A_{J_1}$) and $h_{i_0} < 0$(because $h \in A_{J_2}$), which is not possible and we have proved that Φ is the family of pairwise disjoint universally measurable subsets in $(R^\infty, +, \rho)$.

Note that

$$\text{card}(\Phi) = \text{card}(\{J : J \subseteq N\}) = c.$$

This ends the proof of Theorem 15.1.1. $\square$

Remark 15.1.1 Theorem 15.1.1 gives a strict argument for applications of non-σ-finite Borel measures in infinite-dimensional analysis. Note here that such applications were considered in Chapter 7(cf. interpretations 7.1–7.5).

Remark 15.1.2 The result of Theorem 15.1.1 is belonging to Dougherty (cf. [66, Theorem 2.18.2, p.59). Other interesting investigations in general groups concerning with the Problem 15.1.1, can be found in [66, Chapters 2,5].

15.2. On one question of R.Baker

Let $\mathbb{R}$ be the real line and $\mathbb{R}^\infty$ stand for the space of all real-valued sequences, equipped with the Tychonoff topology (i.e., the product topology).

Let us denote by $\mathcal{B}(\mathbb{R}^\infty)$ the σ-algebra of all Borel subsets in $\mathbb{R}^\infty$. Further, let $\mathcal{R}_1$ be the class of all infinite dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^\infty)$ of the form

$$R = \prod_{i=1}^{\infty}(a_i; b_i), \ -\infty < a_i \leq b_i < +\infty,$$

such that $0 \leq \prod_{i=1}^{\infty}(b_i - a_i) := \lim_{n\to\infty}\prod_{i=1}^{n}(b_i - a_i) < \infty$.

Let τ_1 be the set function on $\mathcal{R}_1$ defined by

$$\tau_1(R) = \prod_{i=1}^{\infty}(b_i - a_i), \ R \in \mathcal{R}_1.$$

R.Baker[2] proved that the functional λ_1, defined by

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^\infty) \to \lambda_1(X) = \inf\{\sum_{j=1}^{\infty} \tau_1(R_j) : R_j \in \mathcal{R}_1 \ \& \ X \subseteq \cup_{j=1}^{\infty} R_j\}),$$

is a translation-invariant Borel measure on $\mathbb{R}^\infty$.

Now, let $\mathcal{R}_2$ be the class of all infinite dimensional rectangles $R \in \mathcal{B}(R^\infty)$ of the form

$$R = \prod_{i=1}^{\infty} R_i, \ R_i \in \mathcal{B}(\mathbb{R}),$$

such that

$$0 \le \prod_{i=1}^{\infty} m(R_i) := \lim_{n \to \infty} \prod_{i=1}^{n} m(R_i) < \infty,$$

where m denotes one-dimensional classical Borel measure on $\mathbb{R}$.

Let τ_2 be the set function on $\mathcal{R}_2$, defined by

$$\tau_2(R) = \prod_{i=1}^{\infty} m(R_i), \ R_i \in \mathcal{R}_2.$$

R.Baker[3] proved that the functional λ_2, defined by

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^\infty) \to \lambda_2(X) = \inf\{\sum_{j=1}^{\infty} \tau_2(R_j) : R_j \in \mathcal{R}_2 \ \& \ X \subseteq \cup_{j=1}^{\infty} R_j\})$$

is a translation-invariant Borel measure on $\mathbb{R}^\infty$ and posed the following question: “We do not know whether or not the measure λ (i.e., λ_2) coincides with the original version (i.e., with the measure λ_1).”

To resolve this question we need some standard notions.

Let μ_1 and μ_2 be two measures defined on the measurable space $(\mathbb{E}, \mathbb{S})$.

Definition 15.2.1 ([54],p 124). We say that μ_1 is absolutely continuous with respect to μ_2, in symbols $\mu_1 \ll \mu_2$, if

$$(\forall X)(X \in \mathbb{S} \ \& \ \mu_2(X) = 0 \to \mu_1(X) = 0).$$

Definition 15.2.2 ([54],p 126). Two measures μ_1 and μ_2 for which both $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$ are called equivalent, in symbols $\mu_1 \equiv \mu_2$.

The answer to above-mentioned question is contained in the following assertion.

Theorem 15.2.1 *The measures λ_1 and λ_2 are not equivalent and $\lambda_2 \ll \lambda_1$.*

Proof. Let show that the measure λ_1 is not absolutely continuous with respect to the measure λ_2.

We set

$$(\forall k)(k \in \mathbb{N} \to \Delta_k = \sum_{i=2}^{\infty} \Delta_k^{(i)}),$$

where $\Delta_k^{(i)} = [3i; 3i + \frac{1}{2^i}]$ for $i \in \mathbb{N}$.

Let consider a Borel set $E = \prod_{k=1}^{\infty} \Delta_k$.

On the one hand, we have

$$\lambda_2(E) = \inf\{\sum_{j=1}^{\infty}\tau_2(R_j) : R_j \in \mathcal{R}_2 \ \&\ E \subseteq \cup_{j=1}^{\infty}R_j\} = \tau_2(\prod_{k=1}^{\infty}\Delta_k) =$$

$$\prod_{k=1}^{\infty} m(\Delta_k) = \lim_{n\to\infty}\prod_{k=1}^{n} m(\Delta_k) = \lim_{n\to\infty}\frac{1}{2^n} = 0.$$

On the other hand, the set E is not covered by the union of any countable family of elements of $\mathcal{R}_1$.

Indeed, assume the contrary and let

$$(R_j)_{j\in\mathbb{N}} = (\prod_{k=1}^{\infty}(a_k^{(j)}; b_k^{(j)}))_{j\in\mathbb{N}}$$

be a countable family of elements of $\mathcal{R}_1$ which covers E.

We have

$$E = \prod_{k\in N}\Delta_k = \prod_{k\in\mathbb{N}}(\sum_{i=2}^{\infty}\Delta_k^{(i)}).$$

For $j \in \mathbb{N}$ we choose $\Delta_j^{(i_j)}$ such that

$$(a_j^{(j)}; b_j^{(j)}) \cap \Delta_j^{(i_j)} = \emptyset.$$

Then $\prod_{j\in\mathbb{N}}\Delta_j^{(i_j)} \subset E$ and

$$(\forall j)(j \in \mathbb{N} \to R_j \cap \prod_{k=1}^{\infty}\Delta_k^{(i_k)} = \emptyset).$$

We obtain a contradiction and thus, E is not covered by the union of any countable family of elements from $\mathcal{R}_1$. Following convention [2], any infimum taken over an empty set of real numbers has the value $+\infty$. It follows that $\lambda_1(E) = \infty$. Thus, we get

$$\lambda_2(E) = 0 < \infty = \lambda_1(E).$$

Now let show that the measure $\lambda_2 \ll \lambda_1$. This fact follows from the inclusion $\mathcal{R}_1 \subset \mathcal{R}_2$. Indeed, if $\lambda_1(X) = 0$, then

$$\lambda_2(X) = \inf\{\sum_{j=1}^{\infty}\tau_2(R_j) : R_j \in \mathcal{R}_2 \ \&\ X \subseteq \cup_{j=1}^{\infty}R_j\}) \le$$

$$\inf\{\sum_{j=1}^{\infty}\tau_2(R_j) : R_j \in \mathcal{R}_1 \ \&\ X \subseteq \cup_{j=1}^{\infty}R_j\}) = \lambda_1(X) = 0.$$

This ends the proof of Theorem 15.2.1. □

In context of R.Baker's measures we state the following

Conjecture 15.2.1 *The function f_E(cf. Chapter 7,p. 118) is measurable for arbitrary $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ and the functional μ defined by*

$$(\forall E)(E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to \mu(E) = \int_{\prod_{k\in\mathbb{N}} S_k} f_E(g)d\overline{\lambda_{\mathbb{N}}}(g))$$

coincides with "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$ constructed in [3].

15.3. Representation of Baker measure in the sense [2] by A. Kharazishvili's measures [87]

Let $\mathbb{R}^{\mathbb{N}}$ be the topological vector space of all real-valued sequences equipped with the Tychonoff topology. Let us denote by $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ the σ-algebra of all Borel subsets in $\mathbb{R}^{\mathbb{N}}$.

Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be sequences of real numbers such that

$$(\forall i)(i \in \mathbb{N} \rightarrow a_i < b_i)$$

and

$$0 < \prod_{k\in\mathbb{N}}(b_i - a_i) < \infty$$

We put

$$A(\Delta,n) = \mathbb{R}_0 \times \cdots \times \mathbb{R}_n \times \prod_{i>n}\Delta_i$$

for $n \in \mathbb{N}$, where

$$(\forall i)(i \in \mathbb{N} \rightarrow \mathbb{R}_i = \mathbb{R} \ \&\ \Delta_i = [a_i;b_i[).$$

We put also

$$\Delta = \prod_{i\in\mathbb{N}}\Delta_i \ \ \& \ \ B_\Delta = \cup_{n\in\mathbb{N}}A(\Delta,n).$$

For an arbitrary natural number $i \in \mathbb{N}$, let consider the classical one-dimensional Borel measure μ_i defined on the space $\mathbb{R}_i = \mathbb{R}$. Let λ_i denote the restriction of the measure μ_i on the interval Δ_i.

For an arbitrary $n \in \mathbb{N}$, let us denote by ν_n the measure defined by

$$\nu_n = \prod_{1\leq i\leq n}\mu_i \times \prod_{i>n}\lambda_i,$$

and by $\overline{\nu}_n$ the Borel measure in the space $\mathbb{R}^{\mathbb{N}}$ defined by

$$(\forall X)(X \in B(\mathbb{R}^{\mathbb{N}}) \rightarrow \overline{\nu}_n(X) = \nu_n(X \cap A(\Delta,n))).$$

Remark 15.3.1 Note that a class of sets

$$\mathcal{A} = \{X \times \prod_{i>n+p}[a_i;b_i[: \ X \in \mathcal{B}(\mathbb{R}^n \times \prod_{i=1}^{n+p}[a_i;b_i[), \ n,p \in \mathbb{N}\}$$

is an algebra of subsets of $A(\Delta,n)$ and the functional λ defined by

$$\lambda(X \times \prod_{i>n+p}[a_i;b_i[) = (\prod_{i=1}^{n}\mu_i \times \prod_{i=n+1}^{n+p}\lambda_i)(X) \times \prod_{i>n+p}(b_i - a_i)$$

is a σ-finite measure on $\mathcal{A}$. By using Charatheodory theorem, this measure has a unique extension on $\mathcal{B}(A(\Delta,n))$, which is denoted by $\prod_{i=1}^{n}\mu_i \times \prod_{i>n}\lambda_i$.

Using the same arguments applied in proofs of Lemma 5.1 and Theorem 5.1, one can prove the following assertions.

Lemma 15.3.2 *For an arbitrary Borel set $X \subseteq \mathbb{R}^{\mathbb{N}}$ there exists a limit*

$$\nu_\Delta(X) = \lim_{n\to\infty} \overline{\nu}_n(X).$$

Moreover, the functional ν_Δ is a nontrivial σ-finite measure defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ which

$$\nu_\Delta(\Delta) = \prod_{k=1}^{\infty}(b_k - a_k).$$

Let G_Δ denotes a group of all admissible translations (in the sense of invariance) of the measure ν_Δ. Then we have

Lemma 15.3.2 *The following conditions are equivalent:*
1) $g = (g_1, g_2, \cdots) \in G_\Delta$;
2) $(g_i)_{i\in\mathbb{N}} \in \ell_1$;

Now let **K** be a class of all positive sequences $(a_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$ such that $0 < \prod_{k\in\mathbb{N}} a_k < \infty$.

Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in \mathbf{K}$. We say that $(a_k)_{k\in\mathbb{N}} \simeq (b_k)_{k\in\mathbb{N}}$ if and only if $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$.

Lemma 15.3.3 *The relation $\simeq$ is equivalence on* **K**.

Proof.

I. Obviously, $(a_k)_{k\in\mathbb{N}} \simeq (a_k)_{k\in\mathbb{N}}$ because $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,a_k]}$.

II. If $(a_k)_{k\in\mathbb{N}} \simeq (b_k)_{k\in\mathbb{N}}$ then $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$. Evidently, $\nu_{\prod_{k\in\mathbb{N}}[0,b_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,a_k]}$ and we deduce that $(b_k)_{k\in\mathbb{N}} \simeq (a_k)_{k\in\mathbb{N}}$.

III. If $(a_k)_{k\in\mathbb{N}} \simeq (b_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}} \simeq (c_k)_{k\in\mathbb{N}}$, then $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$ and $\nu_{\prod_{k\in\mathbb{N}}[0,b_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,c_k]}$, which follows $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,c_k]}$. Hence, we obtain $(a_k)_{k\in\mathbb{N}} \simeq (c_k)_{k\in\mathbb{N}}$. $\square$

Lemma 15.3.4 *Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in$ **K** are not equivalent. Then $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} \perp \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$.*

Proof. Let show that

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}(\prod_{k\in\mathbb{N}}[0,a_k] \cap \prod_{k\in\mathbb{N}}[0,b_k]) = 0.$$

Assume the contrary. Then

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}(\prod_{k\in N}[0,a_k] \cap \prod_{k\in\mathbb{N}}[0,b_k]) > 0$$

and by using metrically transitivity of $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}$ and $\nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$, we conclude that there exists a countable ℓ_1-configuration D^* of $\prod_{k\in\mathbb{N}}[0,a_k] \cap \prod_{k\in\mathbb{N}}[0,b_k]$ such that

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}(B_{\prod_{k\in\mathbb{N}}[0,a_k]} \setminus D^*) = \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}(B_{\prod_{k\in\mathbb{N}}[0,b_k]} \setminus D^*) = 0.$$

Clearly,

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}(\prod_{k\in\mathbb{N}}[0,a_k] \cap \prod_{k\in\mathbb{N}}[0,b_k]) =$$

$$\nu_{\prod_{k\in\mathbb{N}}[0,b_k]}(\prod_{k\in\mathbb{N}}[0,a_k]\cap\prod_{k\in\mathbb{N}}[0,b_k])=\nu_{\prod_{k\in\mathbb{N}}[0,\min\{a_k,b_k\}]}(\prod_{k\in\mathbb{N}}[0,\min\{a_k,b_k\}]).$$

Using the technique, applied in Chapter 8, one can demonstrate that

$$\nu_{\prod_{k\in\mathbb{N}}[0;a_k]}=\nu_{\prod_{k\in\mathbb{N}}[0,b_k]}=\nu_{\prod_{k\in\mathbb{N}}[0;\min\{a_k,b_k\}]}$$

and we obtain a contradiction with the condition

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]}\neq\nu_{\prod_{k\in\mathbb{N}}[0,b_k]},$$

because $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are not equivalent.

This ends the proof of Lemma 15.3.4. $\square$

Definition 15. 3.1 Let $\mathcal{R}_1$ be the class of all infinite dimensional rectangles $R\in\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R=\prod_{k=1}^{\infty}(a_i,b_i),\ -\infty<a_i<b_i<\infty$$

such that $0\le\prod_{k=1}^{\infty}(b_k-a_k)<\infty$.

A translation-invariant Borel measure λ on $\mathbb{R}^{\mathbb{N}}$ is said to be a Baker measure in the sense [2] if

$$\lambda(\prod_{k=1}^{\infty}(a_k,b_k))=\prod_{k=1}^{\infty}(b_k-a_k).$$

Let $\mathcal{F}$ be a class of vector subspaces F in V that the following relation is fulfilled

$$(\forall F)(F\in\mathcal{F}\rightarrow L\cap F=\{\mathbf{0}\}\ \&\ L+F=V).$$

We set $L^{\perp}=\tau(\mathcal{F})$, where τ denotes a global operator of choice. A vector subspace $L^{\perp}$ is said to be a linear complement of the vector space L in V.

Below, by using the structure of A.B. Kharazishvili's measures, we will construct an example of a Baker measure in the sense [2] which, unlike the Baker measure [2] and like the Baker measure[3], gets a numerical value zero on the set N^N.

Theorem 15.3.1 *There exist a Baker measure in the sense [2] which gets a numerical value zero on the set* N^N.

Proof. Let consider the classes of equivalence $(\mathbf{K}_i)_{i\in I}$ of $\mathbf{K}$, generated by the relation of equivalence $\simeq$(cf. Lemma 15.3.3). Let $(a_k^{(i)})_{k\in\mathbb{N}}\in\mathbf{K}_i$ for $i\in I$.

We set $\Delta_i=\prod_{k\in\mathbb{N}}[0,a_k^{(i)}]$, $B_i=B_{\Delta_i}$ and $\mu_i=\nu_{\Delta_i}$.

We put

$$(\forall X)(X\in\mathcal{B}(\mathbb{R}^{\mathbb{N}})\rightarrow\lambda(X)=\sum_{i\in I}\sum_{g\in\ell_1^{\perp}}\mu_i(X-g\cap B_i)),$$

where $\ell_1^{\perp}$ denotes a linear complement of the vector space ℓ_1 in $\mathbb{R}^{\mathbb{N}}$.

Let show that λ is a translation-invariant Borel measure on $\mathbb{R}^{\mathbb{N}}$ such that

$$\lambda(\prod_{k=1}^{\infty}(a_k,b_k))=\prod_{k=1}^{\infty}(b_k-a_k).$$

Indeed:

I. Let $X = \sum_{k\in\mathbb{N}} X_k$, where $X_k \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ for $k \in \mathbb{N}$. Then

$$\lambda(\sum_{k\in\mathbb{N}} X_k) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(\sum_{k\in\mathbb{N}} X_k - g \cap B_i) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(\sum_{k\in\mathbb{N}} (X_k - g) \cap B_i) =$$

$$\sum_{i\in I}\sum_{g\in\ell_1^{\perp}}\sum_{k\in\mathbb{N}} \mu_i(X_k - g \cap B_i) = \sum_{k\in\mathbb{N}}\sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(X_k - g \cap B_i) = \sum_{k\in\mathbb{N}} \lambda(X_k).$$

Hence, the functional λ is σ-additive.

II. Let show that λ is translation-invariant.

Take into account the equality $G_{\Delta} = \ell_1$(cf.Lemma 15.3.2) and the fact that an arbitrary element $h \in$
$mathbbR^N$ is presented in the form $h = h_1 + h_2$, where $h_1 \in \ell_1$ and $h_2 \in \ell_1^{\perp}$, we get

$$\lambda(X+h) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(((X+h_1+h_2)-g)\cap B_i) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((((X+h_2)-g)+h_1)\cap B_i) =$$

$$\sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((((X+h_2)-g))\cap(B_i+h_1)) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(((X+h_2)-g)\cap B_i) =$$

$$\sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(X-(g-h_2)\cap B_i) = \sum_{i\in I}\sum_{g-h_2\in\ell_1^{\perp}} \mu_i(X-(g-h_2)\cap B_i) =$$

$$\sum_{i\in I}\sum_{\overline{g}\in\ell_1^{\perp}} \mu_i(X-\overline{g}\cap B_i) = \lambda(X).$$

III. Let show that

$$\lambda(\prod_{k=1}^{\infty}(a_k,b_k)) = \prod_{k=1}^{\infty}(b_k-a_k).$$

Let K_{i_0} be a class of equivalence of $\mathbf{K}$ such that $(b_k-a_k)_{k\in\mathbb{N}} \in K_{i_0}$. By using translation-invariance of λ, we have

$$\lambda(\prod_{k=1}^{\infty}(a_k,b_k) - (a_k)_{k\in\mathbb{N}}) = \lambda(\prod_{k=1}^{\infty}(0,b_k-a_k)) =$$

$$\sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(\prod_{k=1}^{\infty}(0,b_k-a_k) - g\cap B_i) =$$

$$\mu_{i_0}(\prod_{k=1}^{\infty}(0,b_k-a_k) - (0,0,\cdots)\cap B_{i_0}) = \prod_{k=1}^{\infty}(b_k-a_k).$$

One can demonstrate that if $\prod_{k\in\mathbb{N}}(b_k-a_k) = 0$, then $\lambda(\prod_{k\in\mathbb{N}}(a_k;b_k)) = 0$.

Finally, $\lambda(N^N) = 0$, because $\mu_i(N^N - g\cap B_i) = 0$ for $i \in I$ and $g, \in \ell_1^{\perp}$.

This ends the proof of Theorem 15.3.1. □

In context of Theorem 15.3.1 the following theorem is of some interest.

Theorem 15.3.2 *There exists a translation-invariant Borel measure* λ_1 *on* $\mathbb{R}^{\mathbb{N}}$ *which satisfies the following conditions:*

1) $\lambda_1([0,1]^{\mathbb{N}}) = 1$;
2) There exists a rectangle $\prod_{k=1}^{\infty}(a_k, b_k)$ *with* $0 < \prod_{k=1}^{\infty}(b_k - a_k) < \infty$ *such that*

$$\lambda_1(\prod_{k=1}^{\infty}(a_k, b_k)) = 0.$$

Proof. We put

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to \lambda_1(X) = \sum_{g \in \ell_1^{\perp}} \nu_{[0,1]^N}(X - g \cap B_{[0,1]^N}).$$

I. Let $X = \sum_{k\in\mathbb{N}} X_k$, where $X_k \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ for $k \in \mathbb{N}$. Then

$$\lambda_1(\sum_{k\in\mathbb{N}} X_k) = \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(\sum_{k\in\mathbb{N}} X_k - g \cap B_{[0,1]^N}) = \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(\sum_{k\in\mathbb{N}} (X_k - g) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in\ell_1^{\perp}} \sum_{k\in\mathbb{N}} \nu_{[0,1]^{\mathbb{N}}}(X_k - g \cap B_{[0,1]^{\mathbb{N}}}) = \sum_{k\in\mathbb{N}} \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(X_k - g \cap B_{[0,1]^{\mathbb{N}}}) = \sum_{k\in\mathbb{N}} \lambda_1(X_k).$$

Hence, the functional λ_1 is σ-additive.

II. Let show that λ_1 is translation-invariant.

Took in an account the equality $G_{[0,1]^{\mathbb{N}}} = \ell_1$ and the simple fact that an arbitrary element $h \in \mathbb{R}^{\mathbb{N}}$ is presented in the form $h = h_1 + h_2$, where $h_1 \in \ell_1$ and $h_2 \in \ell_1^{\perp}$, we get

$$\lambda_1(X + h) = \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(((X + h_1 + h_2) - g) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in{\ell_1}^{\perp}} \nu_{[0,1]^{\mathbb{N}}}((((X + h_2) - g) + h_1) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in{\ell_1}^{\perp}} \nu_{[0,1]^N}((((X + h_2) - g)) \cap (B_{[0,1]^N} + h_1)) =$$

$$\sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(((X + h_2) - g) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in{\ell_1}^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(((X - (g - h_2) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g-h_2\in{\ell_1}^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(((X - (g - h_2) \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{\overline{g}\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(X - \overline{g} \cap B_{[0,1]^{\mathbb{N}}}) = \lambda_1(X).$$

III. Let show that

$$\lambda_1([0,1]^{\mathbb{N}}) = 1.$$

We have

$$\lambda_1([0,1]^{\mathbb{N}}) = \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}([0,1]^{\mathbb{N}} - g\cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\nu_{[0,1]^{\mathbb{N}}}([0,1]^{\mathbb{N}} - (0,\cdots,0)\cap B_{[0,1]^{\mathbb{N}}}) = 1.$$

IV. Let show that there exists a rectangle $\prod_{k=1}^{\infty}(a_k;b_k)$ with $0 < \prod_{k=1}^{\infty}(b_k - a_k) < \infty$ such that

$$\lambda_1(\prod_{k=1}^{\infty}(a_k;b_k)) = 0.$$

Indeed, if we set

$$\prod_{k=1}^{\infty}(a_k;b_k) = \prod_{k=1}^{\infty}(0;1+(-1)^k\frac{1}{k+1}),$$

then we will obtain

$$\lambda_1(\prod_{k=1}^{\infty}(0;1+(-1)^k\frac{1}{k+1})) = \sum_{g\in\ell_1^{\perp}} \nu_{[0;1]^{\mathbb{N}}}(\prod_{k=1}^{\infty}(0,1+(-1)^k\frac{1}{k+1}) - g\cap B_{[0,1]^{\mathbb{N}}}) \leq$$

$$\sum_{g\in\ell_1^{\perp}} \nu_{[0;1]^{\mathbb{N}}}(\prod_{k=1}^{\infty}(0,1+(-1)^k\frac{1}{k+1})\cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in\ell_1^{\perp}} (\lim_{n\to\infty} \nu_n(\prod_{k=1}^{\infty}(0;1+(-1)^k\frac{1}{k+1})\cap A([0,1]^{\mathbb{N}},n)) =$$

$$\sum_{g\in\ell_1^{\perp}} (\lim_{n\to\infty} \prod_{k=1}^{n}(1+(-1)^k\frac{1}{k+1}) \times (\prod_{i>n}\lambda_i)(\prod_{i>n}(0;1+(-1)^n\frac{1}{n+1})\cap[0,1]^{\mathbb{N}\setminus\{1,\cdots,n\}}) =$$

$$\sum_{g\in\ell_1^{\perp}} \lim_{n\to\infty} \prod_{k=1}^{n}(1+(-1)^k\frac{1}{k+1}) \times (\prod_{i>n}\lambda_i)(\prod_{i>n}(0,\min\{1;1+(-1)^i\frac{1}{i+1}\})) = 0.$$

This ends the proof of Theorem 15.3.2. $\square$

By using theorems 15.3.1 and 15.3.2, we give

Corollary 15.3.1 R. Baker's measure λ[2] (unlike the classical n-dimensional Borel measure b_n on $\mathbf{R}(\mathrm{n} \geq 1)$), does not have the property of uniqueness in the class of all translation-invariant Borel measures on $\mathbb{R}^{\mathbb{N}}$ which gets a value one on the standard cube.

Remark 15.3.2 Note that the measure λ_1 is absolutely continuous with respect to the measure λ, but the converse relation is not valid (cf. Theorem 15.3.2, condition **2**)). It means that the measure λ and λ_1 are not equivalent. In this context we must say that there does not exist a Borel measure μ on $\mathbb{R}^n$ which is translation-invariant, absolutely continuous with respect to the classical n-dimensional Borel measure b_n, gets a value one on $[0,1]^n$ and $b_n \neq \mu$.

Now let μ be a Haar measure on $[0;1]^{\mathbb{N}}$.

We say that a Borel subset $X \subset \mathbb{R}^{\mathbb{N}}$ is a standard cube null set if

$$(\forall a)(a \in \mathbb{R}^{\mathbb{N}} \to \mu(X + a \cap [0;1]^{\mathbb{N}}) = 0).$$

We denote the class of all standard cube null sets by $\mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$.

Let ν be any Borel measure on $\mathbb{R}^{\mathbb{N}}$. We denote the class of all ν-null sets by $\mathcal{N}.\mathcal{S}.(\nu)$.

Here naturally arises the following:

Problem 15.3.1 Does there exist a quasi-finite translation-invariant Borel measure ν on $\mathbb{R}^{\mathbb{N}}$ such that $\mathcal{N}.\mathcal{S}.(\nu) = \mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$?

The answer on this problem is contained in the following assertion.

Theorem 15.3.3 *The measure* λ_1 *is solution of Problem* 15.3.1*, i.e.,*

$$\mathcal{N}.\mathcal{S}.(\lambda_1) = \mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}).$$

Proof. Let show that

$$\mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}) \subseteq \mathcal{N}.\mathcal{S}.(\lambda_1).$$

We set

$$\mathbb{Z}^{(\mathbb{N})} = \{(z_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} : \mathrm{card}(\{i : z_i \neq 0\}) < \omega\}.$$

For $X \in \mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$ we have

$$\lambda_1(X) = \sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(X - g \cap B_{[0,1]^{\mathbb{N}}}) =$$

$$\sum_{g\in\ell_1^{\perp}} \nu_{[0,1]^{\mathbb{N}}}(X - g \cap \sum_{f\in\mathbb{Z}^{(\mathbb{N})}} ([0,1[^{\mathbb{N}} + f)) =$$

$$\sum_{g\in\ell_1^{\perp}} \sum_{f\in\mathbb{Z}^{(\mathbb{N})}} \nu_{[0,1]^{N}}(X - g \cap ([0,1[^{\mathbb{N}} + f)) =$$

$$\sum_{g\in\ell_1^{\perp}} \sum_{f\in\mathbb{Z}^{(\mathbb{N})}} \nu_{[0,1]^{N}}(X - g - f \cap [0,1[^{\mathbb{N}}) =$$

$$\sum_{g\in\ell_1^{\perp}} \sum_{f\in\mathbb{Z}^{(\mathbb{N})}} \mu(X - g - f \cap [0,1[^{\mathbb{N}}) = 0.$$

The last relation means that $X \in \mathcal{N}.\mathcal{S}.(\lambda_1)$.

Now let $X \in \mathcal{N}.\mathcal{S}.(\lambda_1)$. Then we have

$$\mu(X + a \cap [0,1]^{\mathbb{N}}) = \lambda_1(X + a \cap [0,1]^{\mathbb{N}}) \leq \lambda_1(X + a) = \lambda_1(X) = 0.$$

The last relation means that $X \in \mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$.

This ends the proof of Theorem 15.3.3. □

Remark 15.3.3 If we consider R.Baker's measure λ, constructed in [2] or [3], then we observe that

$$\mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}) \setminus \mathcal{N}.\mathcal{S}.(\lambda) \neq \emptyset.$$

Indeed, following Theorem 15.3.2, there exists a rectangle $\prod_{k\in\mathbb{N}}(a_k;b_k)$ such that $\lambda(\prod_{k\in\mathbb{N}}(a_k;b_k)) > 0$ and $\lambda_1(\prod_{k\in\mathbb{N}}(a_k;b_k)) = 0$.

It can be proved that the following inclusions

$$\mathcal{N}.\mathcal{S}.(\lambda(1)) \subset \mathcal{N}.\mathcal{S}.(\lambda(2)) \subset \mathcal{N}.\mathcal{S}.(\lambda_1)(= \mathcal{S}.\mathcal{C}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}))$$

are strict, where by $\lambda(1)$ and $\lambda(2)$ we have denoted "Lebesgue measures" on $\mathbb{R}^{\mathbb{N}}$ constructed in [2] and [3], respectively.

Following [70], a set $X \subset \mathbb{R}^{\mathbb{N}}$ is called **shy** if it is a subset of a Borel set X' for which $\mu(X'+v) = 0$ for every $v \in \mathbb{R}^{\mathbb{N}}$ and some μ such that $\mu(K) = \mu(B)$ for some compact K.

The class of all shy sets in $\mathbb{R}^{\mathbb{N}}$ we denote by $\mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$.

In context of Theorem 15.3.3 the following assertion is of some interest.

Theorem 15.3.4 *There does not exist a quasi-finite translation-invariant Borel measure* ν *on* $\mathbb{R}^{\mathbb{N}}$ *such that*

$$\mathcal{N}.\mathcal{S}.(\nu) = \mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}).$$

Proof. Assume the contrary and let ν be a such measure. Since ν is a quasi-finite measure there exists a Borel subset $U \subset \mathbb{R}^{\mathbb{N}}$ such that $0 < \nu(U) < \infty$. From inner regularity of Borel probability measures on $\mathbb{R}^{\mathbb{N}}$ there exists a compact set $K \subseteq U$ such that $0 < \nu(K) \leq \nu(U)$. Since K is shy set (cf.[70], Fact 8, p.225), we conclude that $K \in \mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$, which means that

$$\mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}) \setminus \mathcal{N}.\mathcal{S}.(\nu) \neq \emptyset.$$

This ends the proof of Theorem 15.3.4. □

Remark 15.3.4 Let μ_n be an n-dimensional standard Borel measure defined on the n-dimensional vector subspace $L \subset \mathbb{R}^{\mathbb{N}}$, where $n \in \mathbb{N}$. Let $L^{\perp}$ be a linear complement of L in $\mathbb{R}^{\mathbb{N}}$. Then the functional λ_n defined by

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \rightarrow \lambda_n(X) = \sum_{g \in L^{\perp}} \mu_n(X - g \cap L))$$

is called an n-dimensional "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$.

Since the n-dimensional "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$ is translation-invariant quasi-finite Borel measure, we can easily establish that $\mathcal{N}.\mathcal{S}.(\lambda_n) \subset \mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$ for $n \in \mathbb{N}$.

Note that non-σ-finite quasi-finite translation-invariant Borel measures were not under consideration in mathematics. Below we present another interesting (in our sense) construction of such measures in some metrizable (with complete invariant metric) topological vector spaces and indicate their connections with geometry in corresponding spaces.

Proposition 15.3.1 *Let* $(B, ||\cdot||)$ *be a non-empty Banach space. Then there exists a quasi-finite translation invariant Borel measure* λ *(which is not* σ*-finite) in* B *such that for an arbitrary broken line* $A_0A_1\cdots A_p$ *defined by*

$$A_0A_1\cdots A_p \equiv [A_0A_1[\cup\cdots\cup[A_{p-1}A_p]$$

we have

$$\lambda(A_0A_1\cdots A_p) = \sum_{k=0}^{p-1} ||A_{k+1} - A_k||.$$

Proof. Let B_0 be a unit sphere in B, i.e.,

$$B_0 = \{h : h \in B \; \& \; ||h|| = 1\}.$$

For $h_1, h_2 \in B_0$ we say that $h_1 \approx h_2$ if and only if $h_1 \parallel h_2$. It is clear that $\approx$ is relation of equivalence on B_0. Let $(K_i)_{i\in I}$ be a family of all equivalent classes in B_0. Let $(a_i)_{i\in I}$ be a family of elements of B_0 such that $a_i \in K_i$ for $i \in I$.

Let Γ_i be a line which is parallel to the vector a_i for $i \in I$ and which contains a zero of B. Let $\Gamma_i^{\perp}$ be a linear complement of the vector space Γ_i for $i \in I$. Let μ_i be the one-dimensional classical Borel measure on Γ_i which

$$\mu_i(\{t \times a_i : 0 \le t \le 1\}) = 1.$$

We put

$$(\forall X)(X \in \mathcal{B}(B) \rightarrow \lambda(X) = \sum_{i\in I} \sum_{g \in \Gamma_i^{\perp}} \mu_i(X - g \cap \Gamma_i)).$$

Now one can easily check that all conditions formulated for the measure λ in Proposition 15.3.1, are satisfied. □

Proposition 15.3.2 *Let $\mathbb{R}^3$ be a three dimensional Euclidean vector space. Then there exists a quasi-finite translation invariant Borel measure μ (which is not σ-finite) in $\mathbb{R}^3$ such that for an arbitrary Polyhedron $P \subset \mathbb{R}^3$ we have*

$$\mu(B(P)) = S(B(P)),$$

where $B(P)$ denotes the surface of the polyhedron P and $S(B(P))$ denotes the surface area of P.

Proof. Let B_0 be a unit sphere in $\mathbb{R}^3$, i.e.,

$$B_0 = \{h : h \in \mathbb{R}^3 \ \& \ ||h|| = 1\}.$$

For $h_1, h_2 \in B_0$ we say that $h_1 \approx h_2$ if and only if $h_1 \parallel h_2$. It is clear that $\approx$ is the relation of equivalence on B_0. Let $(K_i)_{i\in I}$ be a family of all equivalent classes in B_0. Let $(a_i)_{i\in I}$ be a family of elements of B_0 such that $a_i \in K_i$ for $i \in I$.

Let Γ_i be a plane which has a normal a_i for $i \in I$ and which contains a zero of $\mathbb{R}^3$. Let $\Gamma_i^{\perp}$ be a linear complement of the plane Γ_i for $i \in I$. Let μ_i be a standard two-dimensional classical Borel measure on Γ_i.

We put

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^3) \rightarrow \mu(X) = \sum_{i\in I} \sum_{g \in \Gamma_i^{\perp}} \mu_i(X - g \cap \Gamma_i)).$$

Now one can easily check that all conditions formulated for the measure μ in Proposition 15.3.2 are satisfied. □

15.4. "Gaussian Measure" on R^N

Let $\mathbb{R}^{\mathbb{N}}$ be the topological vector space of all real-valued sequences equipped with the Tychonoff topology. Let us denote by $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ the σ-algebra of all Borel subsets in $\mathbb{R}^{\mathbb{N}}$.

Let $\Delta = \prod_{i\in\mathbb{N}}(a_i, b_i)$ be an infinite-dimensional rectangle such that

$$0 < \prod_{i\in\mathbb{N}}(\Phi(b_i) - \Phi(a_i)) < \infty,$$

where Φ denotes the distribution function of the one-dimensional standard Gaussian random variable (i.e., $\Phi(x)=\int_{\infty}^{x}e^{-\frac{t^2}{2}dt}$ for $x\in\mathbb{R}$).

We put

$$A(\Delta,n)=\mathbf{R}_0\times\cdots\times\mathbf{R}_n\times\prod_{i>n}\Delta_i\ ,$$

for $n\in N$, where

$$(\forall i)(i\in N\rightarrow\mathbf{R}_i=\mathbf{R}\ \&\ \Delta_i=[a_i;b_i[).$$

We put also

$$\Delta=\prod_{i\in N}\Delta_i.$$

For an arbitrary natural number $i\in\mathbb{N}$ we denote by μ_i the one-dimensional standard Gaussian measure on $\mathbb{R}_i=\mathbb{R}$. Let λ_i denote the restriction of μ_i on the interval Δ_i.

For an arbitrary $n\in N$, let us denote by ν_n the measure defined by

$$\nu_n=\prod_{1\leq i\leq n}\mu_i\times\prod_{i>n}\lambda_i,$$

and by $\overline{\nu}_n$ the Borel measure in the space $\mathbb{R}^{\mathbb{N}}$ defined by

$$(\forall X)(X\in\mathcal{B}(\mathbb{R}^{\mathbb{N}})\rightarrow\overline{\nu}_n(X)=\nu_n(X\cap A(\Delta,n))).$$

Remark 15.4.1 Note that a class of sets

$$\mathcal{A}=\{X\times\prod_{i>n+p}[a_i;b_i[:\ X\in\mathcal{B}(\mathbb{R}^n\times\prod_{i=1}^{n+p}[a_i;b_i[),\ n,p\in\mathbb{N}\}$$

is an algebra of subsets of $A(\Delta,n)$ and the functional λ defined by

$$\lambda(X\times\prod_{i>n+p}[a_i;b_i[)=\prod_{i=1}^{n}\mu_i\times\prod_{i=n+1}^{n+p}\lambda_i(X)\times\prod_{i>n+p}(\Phi(b_i)-\Phi(a_i))$$

is a finite measure on $\mathcal{A}$. By using Charatheodory theorem, this measure has a unique extension on $\mathcal{B}(A(\Delta,n))$, which is denoted by $\prod_{i=1}^{n}\mu_i\times\prod_{i>n}\lambda_i$.

Using the same arguments applied in the proof of Lemma 5.1 and Theorem 5.1, one can prove the following assertions.

Lemma 15.4.1 *For an arbitrary Borel set $X\subseteq\mathbb{R}^{\mathbb{N}}$ there exists a limit*

$$\nu_\Delta(X)=\lim_{n\to\infty}\overline{\nu}_n(X).$$

Moreover, the functional ν_Δ coincides with the Standard Gaussian Borel measure Γ defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ which

$$\nu_\Delta(\Delta)=\prod_{k=1}^{\infty}(\Phi(b_k)-\Phi(a_k)).$$

Let G_Δ denotes a group of all admissible translations (in the sense of quasiinvariance). Then Kakutani's well-known result (cf. Corollary 4.4) admits the following equivalent formulation.

Lemma 15.4.2 *The following conditions are equivalent:*

1) $g = (g_1, g_2, \cdots) \in G_\Delta$;

2) $(g_i)_{i\in N} \in \ell_2$;

One can easily prove the following Lemma.

Lemma 15.4.3 *Let* $(g_k)_{k\in N} \notin \ell_2$. *Then* $\nu_\Delta \perp \nu_\Delta^{(g)}$, *where*

$$(\forall X)(X \in B(\mathbb{R}^{\mathbb{N}}) \to \nu_\Delta^{(g)}(X) = \nu_\Delta(X - g \cap B_\Delta)$$

and

$$B_\Delta = \cup_{n\in\mathbb{N}} A(\Delta, n).$$

Let introduce the notion of "Gaussian measure" on R^N.

Definition 15.4.1 A Borel measure μ on $\mathbb{R}^{\mathbb{N}}$ is called a "Gaussian measure" on $\mathbb{R}^{\mathbb{N}}$, if for an arbitrary infinite dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{k=1}^{\infty}(a_i, b_i), \ -\infty < a_i < b_i < \infty,$$

which

$$0 \le \prod_{k=1}^{\infty}(\Phi(b_k) - \Phi(a_k)) := \lim_{n\to\infty} \prod_{k=1}^{n}(\Phi(b_k) - \Phi(a_k)) < \infty,$$

we have

$$\mu(R) = \prod_{k=1}^{\infty}(\Phi(b_k) - \Phi(a_k))$$

and

$$\mu(R + c) = \mu(R),$$

for all $c \in \ell_2^{\perp}$, where $\ell_2^{\perp}$ denotes a linear complement of ℓ_2.

Theorem 15.4.2 *There exists a "Gaussian measure" on* R^N *and it is translation-quasiinvariant.*

Proof. We put

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to \lambda(X) = \sum_{g\in\ell_2^{\perp}} \Gamma(X - g).$$

I. Let $X = \sum_{k\in\mathbb{N}} X_k$, where $X_k \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ for $k \in \mathbb{N}$. Then

$$\lambda(\sum_{k\in\mathbb{N}} X_k) = \sum_{g\in\ell_2^{\perp}} \Gamma(\sum_{k\in\mathbb{N}} X_k - g) = \sum_{g\in\ell_2^{\perp}} \Gamma(\sum_{k\in\mathbb{N}}(X_k - g)) =$$

$$\sum_{g\in\ell_2^{\perp}} \sum_{k\in\mathbb{N}} \Gamma(X_k - g) = \sum_{k\in\mathbb{N}} \sum_{g\in\ell_2^{\perp}} \Gamma(X_k - g) = \sum_{k\in\mathbb{N}} \lambda(X_k).$$

Hence, the functional λ is σ-additive.

II. Let show that for an arbitrary infinite dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{k=1}^{\infty}(a_i, b_i), \ -\infty < a_i < b_i < \infty,$$

which $0 \le \prod_{k=1}^{\infty}(\Phi(b_k) - \Phi(a_k)) < \infty$, the equality

$$\lambda(R) = \prod_{k=1}^{\infty}(\Phi(b_k) - \Phi(a_k))$$

holds.

Indeed, we have

$$\lambda(R) = \sum_{g \in \ell_2^{\perp}} \Gamma(R - g) =$$

$$\Gamma(R - (0, \cdots, 0)) = \Gamma(R) =$$

$$\Gamma(\prod_{n \in \mathbb{N}} (\prod_{k=1}^{n}(a_k, b_k) \times \mathbb{R}^{\mathbb{N} \setminus \{1, \cdots, n\}}) =$$

$$\lim_{n \to \infty} \prod_{k=1}^{n}(\Phi(b_k) - \Phi(a_k)) = \prod_{k=1}^{\infty}(\Phi(b_k) - \Phi(a_k)).$$

III. Let $c \in \ell_2^{\perp}$. Then we will have

$$\lambda(R + c) = \sum_{g \in \ell_2^{\perp}} \Gamma(R + c - g) =$$

$$\sum_{g \in \ell_2^{\perp}} \Gamma(R - (g - c)) = \sum_{\overline{g} \in \ell_2^{\perp}} \Gamma(R - \overline{g}) = \lambda(R).$$

IV. Let show that λ is translation-quasiinvariant.

Take into account that a group of all admissible translations (in the sense of quasiinvariance) G_{Δ} coincides with ℓ_2 and that an arbitrary element $h \in \mathbb{R}^{\mathbb{N}}$ is presented in the form $h = h_1 + h_2$, where $h_1 \in \ell_2$ and $h_2 \in \ell_2^{\perp}$, we get

$$\lambda(X + h) > 0 \Longleftrightarrow \sum_{g \in \ell_2^{\perp}} \Gamma(((X + h_1 + h_2) - g)) > 0 \Longleftrightarrow$$

$$\sum_{g \in \ell_2{}^{\perp}} \Gamma(((X + h_2) - g) + h_1) > 0 \Longleftrightarrow$$

$$\sum_{g \in \ell_2{}^{\perp}} \Gamma((X + h_2) - g) > 0 \Longleftrightarrow$$

$$\sum_{g \in \ell_2{}^{\perp}} \Gamma(X - (g - h_2)) > 0 \Longleftrightarrow$$

$$\sum_{g-h_2 \in \ell_2{}^{\perp}} \Gamma(X-(g-h_2)) > 0 \Longleftrightarrow$$
$$\sum_{\overline{g} \in \ell_2^{\perp}} \Gamma(X-\overline{g}) > 0 \Longleftrightarrow \lambda(X) > 0.$$

This ends the proof of Theorem 15.4.2. □

Remark 15.4.1 Note that the "Gaussian measure" λ on $\mathbb{R}^{\mathbb{N}}$ (like the "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$(cf.[2],[3])) is not σ-finite.

Remark 15.4.2 The role of the measure ν_Δ for construction of the "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$[2] is same as the role of the standard Gaussian measure Γ for construction of the "Gaussian measure" on R^N. By this reason, A.B. Kharazishvili's measure ν_Δ can be considered as the standard Lebesgue measure on $\mathbb{R}^{\mathbb{N}}$.

We say that a Borel subset $X \subset \mathbb{R}^{\mathbb{N}}$ is a standard Gaussian null set if

$$(\forall a)(a \in \mathbb{R}^{\mathbb{N}} \rightarrow \Gamma(X+a)=0).$$

We denote the class of all standard Gaussian null sets by $\mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$.

Here naturally arises the following:

Problem 15.4.1 Does there exist a quasi-finite translation-quasiinvariant Borel measure ν on $\mathbb{R}^{\mathbb{N}}$ such that $\mathcal{N}.\mathcal{S}.(\nu)=\mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$?

The answer on this problem is contained in the following assertion.

Theorem 15.4.3 *The "Gaussian measure" on $\mathbb{R}^{\mathbb{N}}$ is the solution of Problem* 15.4.1*, i.e.,*

$$\mathcal{N}.\mathcal{S}.(\lambda)=\mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}).$$

Proof. Let show that

$$\mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}}) \subseteq \mathcal{N}.\mathcal{S}.(\lambda).$$

For $X \in \mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$ we have

$$\lambda(X)=\sum_{g \in \ell_2^{\perp}} \Gamma(X-g)=0.$$

The last relation means that $X \in \mathcal{N}.\mathcal{S}.(\lambda)$.

Now let $X \in \mathcal{N}.\mathcal{S}.(\lambda)$. Then, for an arbitrary $a \in \mathbb{R}$, we have

$$\lambda(X)=0 \Longleftrightarrow \lambda(X+a)=0 \Longrightarrow \Gamma(X+a-(0,\cdots))=0,$$

which means that $X \in \mathcal{S}.\mathcal{G}.\mathcal{N}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$.

This ends the proof of Theorem 15.4.3. □

Remark 15.4.3 Let γ_n be the n-dimensional Gaussian measure defined on the n-dimensional vector subspace $L \subset \mathbb{R}^{\mathbb{N}}$. Let $L^{\perp}$ be a linear complement of L in $\mathbb{R}^{\mathbb{N}}$. Then the functional Γ_n defined by

$$(\forall X)(X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \rightarrow \Gamma_n(X)=\sum_{g \in L^{\perp}} \gamma_n(X-g))$$

is called an n-dimensional "Gaussian measure" on $\mathbb{R}^{\mathbb{N}}$.

Note that the n-dimensional "Gaussian measure" and the n-dimensional "Lebesgue measure" on $\mathbb{R}^{\mathbb{N}}$ (both defined by any n-dimensional vector subspace $L \subset \mathbb{R}^{\mathbb{N}}$) are equivalent, which follows $\mathcal{N}.\mathcal{S}.(\Gamma_n)=\mathcal{N}.\mathcal{S}.(\lambda_n)$ $(n \in \mathbb{N})$. Following Remark 15.3.3, we deduce that $\mathcal{N}.\mathcal{S}.(\Gamma_n) \subset \mathcal{S}.\mathcal{S}.(\mathbb{R}^{\mathbb{N}})$ for $n \in \mathbb{N}$.

15.5. On one question of P.Komjath

The present section is devoted to investigation of P. Komjath's well-known question, which is formulated as follows.

Question 15.5.1 ([160], Question 1.1, p.2) Suppose that every set of size $\aleph_1$ has Lebesgue measure zero. Does it follow that the union of any set of $\aleph_1$ lines in the plane has Lebesgue measure zero?

P.Komjath's Axiom, denoted by (KA), is the following statement:

every set of size $\aleph_1$ has Lebesgue measure zero.

Following Question 15.5.1, the principal object of this axiom is to settle the following problem:

Does the union of any set of $\aleph_1$ lines in the plane have Lebesgue measure zero?

We have the following assertion.

Theorem 15.5.1. *In the system of axioms (ZFC) & (MA)(cf. Chapter 12) the following three sentences*

1) $\aleph_1 < 2^{\aleph_0}$;

2) (KA);

3) For an arbitrary natural number $n \geq 1$, the union of any set of $\aleph_1$ $n-1$-dimensional planes in $\mathbb{R}^n$ has Lebesgue measure zero,

are equivalent.

Proof. $1) \rightarrow 2)$. Let $X = \{x_i : i \in I\}$ be a subset of $\mathbb{R}$, where $\mathrm{card}(I) = \aleph_1$. Since $\aleph_1 < 2^{\aleph_0}$, using the result of Lemma 12.2, we conclude that $\ell_1(X) = 0$ and the validity of the implication $1) \rightarrow 2)$ is proved.

$2) \rightarrow 3)$. Let $(\Gamma_i)_{i\in I}$ be any set of $\aleph_1$ $n-1$-dimensional planes in $\mathbb{R}^n$.

Since every set of size $\aleph_1$ in $\mathbb{R}$ has Lebesgue measure zero, we conclude that $\mathrm{card}(\mathbb{R}) > \aleph_1$. Using again the result of Lemma 12.2, we conclude that

$$\ell_n(\cup_{i\in I}\Gamma_i) = 0$$

and the validity of the implication $2) \rightarrow 3)$ is proved.

$3) \rightarrow 1)$. Let X be any subset of $\mathbb{R}$ of size $\aleph_1$. We put

$$Y = \cup_{x\in X}\{x\} \times \mathbb{R}.$$

Applying condition $3)$ for $n = 2$, we have $\ell_2(Y) = 0$.

Since $Y_0 = \cup_{x\in X}\{x\} \times [0,1] \subset Y$, we conclude that $\ell_2(Y_0) = 0$. Applying Fubini theorem, we have

$$\ell_2(Y_0) = \ell_1(X) \times \ell_1([0,1]) = \ell_1(X).$$

Finally, we obtain

$$\ell_1(X) = \ell_2(Y_0) \leq \ell_2(Y) = 0.$$

Since every set of size $\aleph_1$ in $\mathbb{R}$ has Lebesgue measure zero, we conclude that $\mathrm{card}(\mathbb{R}) > \aleph_1$ and the validity of the implication $3) \rightarrow 1)$ is proved.

This ends the proof of Theorem 15.5.1. $\square$

Remark 15.5.1. Gödel showed that theory (ZFC) & (CH) (equivalently, theory (ZFC) & (MA) &(CH)) is consistent, since it is valid in the Constructible Universe(cf. [25],p 21). Martin and Solovay proved that statement (MA) & $(\neg CH)$ is consistent with (ZFC)(cf.[25],p 33). Summarizing above-mentioned facts and applying the result of Theorem 15.5.1, we deduce that each of statements

(MA) & (KA);

(MA) & $\neg(KA)$;

is consistent with theory (ZFC).

In particular, we deduce that the system of axioms (ZFC) & (MA) & (KA) is consistent if (ZFC) is consistent. Moreover, theories (ZFC) & (MA) & (KA) and (ZFC) & (MA) & $\neg(CH)$ are equivalent.

By using the result of Theorem 15.5.1, we get

Corollary 15.5.1. *In the system of axioms*

$$(ZFC) \ \& \ (MA) \ \& \ (KA)$$

the union of any set of $\aleph_1$ *lines in the plane has Lebesgue measure zero.*

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